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ScienceDirect
Topology and its Applications 154 (2007) 215–240
Topology and its Applications
www.elsevier.com/locate/topol

Large entire cross-sections of second category sets in $\mathbb{R}^{n+1}$

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Received 11 November 2005; accepted 11 April 2006

Abstract

By the Kuratowski–Ulam theorem, if $A \subseteq \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ is a Borel set which has second category intersection with every ball (i.e., is "everywhere second category"), then there is a $y \in \mathbb{R}$ such that the section $A \cap (\mathbb{R}^n \times \{y\})$ is everywhere second category in $\mathbb{R}^n \times \{y\}$. If A is not Borel, then there may not exist a large cross-section through A, even if the section does not have to be flat. For example, a variation on a result of T. Bartoszynski and L. Halbeisen shows that there is an everywhere second category set $A \subseteq \mathbb{R}^{n+1}$ such that for any polynomial p in n variables, $A \cap \operatorname{graph}(p)$ is finite. It is a classical result that under the Continuum Hypothesis, there is an everywhere second category set L in $\mathbb{R}^{n+1}$ which has only countably many points in any first category set. In particular, $L \cap \operatorname{graph}(f)$ is countable for any continuous function $f : \mathbb{R}^n \to \mathbb{R}$. We prove that it is relatively consistent with ZFC that for any everywhere second category set A in $\mathbb{R}^{n+1}$, there is a function $f : \mathbb{R}^n \to \mathbb{R}$ which is the restriction to $\mathbb{R}^n$ of an entire function on $\mathbb{C}^n$ and is such that, relative to $\operatorname{graph}(f)$, the set $A \cap \operatorname{graph}(f)$ is everywhere second category. For any collection of less than $2^{\aleph_0}$ sets A, the function f can be chosen to work for all sets A in the collection simultaneously. Moreover, given a nonnegative integer k, a function $g : \mathbb{R}^n \to \mathbb{R}$ of class C^k and a positive continuous function $\varepsilon : \mathbb{R}^n \to \mathbb{R}$, we may choose f so that for all multiindices α of order at most k and for all $x \in \mathbb{R}^n$, $|D^\alpha f(x) - D^\alpha g(x)| < \varepsilon(x)$. The method builds on fundamental work of K. Ciesielski and S. Shelah which provides, for everywhere second category sets in $2^\omega \times 2^\omega$, large sections which are the graphs of homeomorphisms of 2^ω. K. Ciesielski and T. Natkaniec adapted the Ciesielski–Shelah result for subsets of $\mathbb{R} \times \mathbb{R}$ and proved the existence in this setting of large sections which are increasing homeomorphisms of $\mathbb{R}$. The technique used in this paper extends to functions of several variables an approach developed for functions of a single variable in previous related work of the author.

MSC: primary 03E35; secondary 30E10

Keywords: Everywhere second category; Entire function; Several complex variables; Oracle-cc forcing; Carleman's theorem; Hoischen

1. Introduction

Recall that when X is a Polish space (i.e., a complete separable metric space), a set $A \subseteq X$ is called nowhere dense if the closure of A has empty interior. A is said to be of first category if $A = \bigcup_{n=1}^{\infty} A_n$, where each A_n is nowhere dense. A is said to be of second category if A is not of first category. A is *everywhere second category* if $A \cap B$ is second category for every ball B, or equivalently, if $A \cap B \neq \emptyset$ for every second category Borel set B. The

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[1] Research supported by NSERC. The author thanks F.D. Tall and the Department of Mathematics at the University of Toronto for their hospitality during the academic year 2003/2004.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2006.04.009

Kuratowski–Ulam theorem ([10], see also [11, Chapter 15]) states in particular that if X and Y are Polish spaces, then for each Borel set $A \subseteq X \times Y$, A is first category if and only if for all but a first category set of $c \in Y$, the section $A \cap (X \times \{c\})$ is first category in $X \times \{c\}$. By taking complements, this can be rephrased as saying that for each Borel set $A \subseteq X \times Y$, A is everywhere second category if and only if for all but a first category set of $c \in Y$, the section $A \cap (X \times \{c\})$ is everywhere second category in X. If A is not Borel, then this theorem can fail dramatically. For example, using the Axiom of Choice, it is easy to construct sets $A \subseteq \mathbb{R}^{n+1}$, n a positive integer, such that A is everywhere second category but no two points of A have a coordinate in common, so that on any hyperplane perpendicular to one of the coordinate axes, A has at most one point. (The set A from the proof of Proposition 1.1 below is one example.) A natural attempt at an alternative to the Kuratowski–Ulam theorem for non-Borel subsets of $\mathbb{R}^{n+1} \cong \mathbb{R}^n \times \mathbb{R}$ involves replacing the horizontal sections $A \cap (\mathbb{R}^n \times \{c\})$ by the sections of A by vertical translates of the graph of a fixed function $f : \mathbb{R}^n \to \mathbb{R}$. So we consider sets of the form $A \cap (f + c)$, where $f + c$ is the function $x \mapsto f(x) + c$, which we identify with its graph. In the Kuratowski–Ulam theorem, $f \equiv 0$. If f is a polynomial, then this attempt fails badly as shown by the following proposition which is a variation on known results.

Proposition 1.1. (Cf. [1].) *There is an everywhere second category set $A \subseteq \mathbb{R}^{n+1}$ such that for every polynomial function $f : \mathbb{R}^n \to \mathbb{R}$, $A \cap f$ is finite.*

Proof. We use the idea of the proof of [4, Theorem 9]. Let $B \subseteq \mathbb{R}$ be a transcendence base for $\mathbb{R}$ over $\mathbb{Q}$ which has nonempty intersection with every uncountable Borel set. Let $\langle A_\alpha \colon \alpha < \mathfrak{c}\rangle$ be a list of all second category Borel sets in $\mathbb{R}^{n+1}$. ($\mathfrak{c}$ as usual denotes the cardinality of $\mathbb{R}$.) Inductively choose points $x_\alpha \in A_\alpha$ so that their coordinates are distinct elements of B. (At stage α, for $i < n$ inductively choose

$$x_\alpha(i) \in B \setminus \bigl(\{x_\alpha(j)\colon\ j < i\} \cup \{x_\beta(k)\colon\ \beta < \alpha,\ k \leqslant n\}\bigr) \tag{$*$}$$

so that the section

$$A_\alpha \cap \bigl\{y \in \mathbb{R}^{n+1}\colon\ y(j) = x_\alpha(j),\ j \leqslant i\bigr\}$$

is second category in $\{y \in \mathbb{R}^{n+1}\colon\ y(j) = x_\alpha(j),\ j \leqslant i\}$ (use the Kuratowski–Ulam theorem). Then pick $x_\alpha(n)$ so that $(*)$ holds for $i = n$ and the point $x_\alpha = (x_\alpha(0), \ldots, x_\alpha(n))$ belongs to A_α.)

Take $A = \{x_\alpha\colon\ \alpha < \mathfrak{c}\}$. Now suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a polynomial. The coefficients of f are algebraic over $\mathbb{Q}(B')$ for some finite $B' \subseteq B$. For any α for which the coordinates of x_α are all outside B', we have $x_\alpha \notin f$ since otherwise $f(x_\alpha(0), \ldots, x_\alpha(n-1)) = x_\alpha(n)$ contradicts the algebraic independence of the elements of B. $\square$

Even if we consider the sections $A \cap (f + c)$ determined by an analytic function f, there will always be (non-Borel) everywhere second category sets $A \subseteq \mathbb{R}^{n+1}$ for which the set of c giving a small (even empty) section is large. This is shown in the next proposition which adapts arguments from [4] to a multivariate context.

Proposition 1.2. (Cf. [4, Theorem 1 and Corollary 2].) *There is a set $A \subseteq \mathbb{R}$ intersecting every uncountable Borel set such that for any analytic function $f : \mathbb{R}^n \to \mathbb{R}$, the set $\{c \in \mathbb{R}\colon\ A^{n+1} \cap (f + c) = \emptyset\}$ intersects every uncountable Borel set.*

(analytic = expandable in a power series in a neighborhood of each point)

Note. Using the Kuratowski–Ulam theorem, it is easy to verify that for such a set A, A^{n+1} is everywhere second category in $\mathbb{R}^{n+1}$.

Proof. Let $\langle K_\alpha\colon\ \alpha < \mathfrak{c}\rangle$ list the uncountable Borel subsets of $\mathbb{R}$. Let $\langle f_\alpha\colon\ \alpha < \mathfrak{c}\rangle$ list the analytic functions $\mathbb{R}^n \to \mathbb{R}$. We shall have $A = \{x_\alpha\colon\ \alpha < \mathfrak{c}\}$ and the points x_α will be chosen inductively, along with points c_α, $\alpha < \mathfrak{c}$, so that the following properties are satisfied.

(1) $x_\alpha \in K_\alpha$ and $c_\alpha \in K_\alpha$.
(2) x_α is not a solution to any equation of the form

$$(f_\beta + c_\gamma)(y_0, \ldots, y_{n-1}) = y_n,$$

where $\beta \leqslant \gamma < \alpha$ and for each $i \leqslant n$, y_i is either the variable x or a constant chosen from $\{x_\beta\colon \beta < \alpha\}$.

(Note that the variable x need not appear in the equation. In this case we are simply saying that the equality fails.)

(3) There are no identities of the form

$$(f_\beta + c_\gamma)(y_0, \ldots, y_{n-1}) \equiv y_n,$$

where $\beta \leqslant \gamma \leqslant \alpha$ and for each $i \leqslant n$, y_i is either the variable x or a constant chosen from $\{x_\beta\colon \beta \leqslant \alpha\}$.

At stage α, first pick x_α then c_α. Note that part (3) of the inductive hypothesis implies that the equations in (2) are not satisfied identically. Since the left- and right-hand sides are analytic functions of a single variable x, each of these equations has only countably many solutions. Consider one of the identities in (3) with $\gamma < \alpha$. If there are uncountably many values of x_α for which the identity holds, then for each fixed value of x, the equation $(f_\beta + c_\gamma)(y_0, \ldots, y_{n-1}) = y_n$, with x_α as the variable, has uncountably many solutions and hence holds identically. In particular, it holds if we replace x_α by the fixed value of x. Hence, we get a new identity from the given one by replacing all occurrences of x_α by x's. But this contradicts part (3) of the induction hypothesis. Hence, each of the identities in (3) with $\gamma < \alpha$ can hold for only countably many values of x_α. Since there are less than $\mathfrak{c}$ such identities and less that $\mathfrak{c}$ equations of the form given in (2), there is no difficulty choosing x_α so that (1), (2) and the cases of (3) with $\gamma < \alpha$ are satisfied. For the cases of (3) where $\gamma = \alpha$, note that each identity of the given form can only hold for a single value of c_α. Since there are less than $\mathfrak{c}$ identities of the given form, we can choose $c_\alpha \in K_\alpha$ so that they all fail.

Now let $A = \{x_\alpha\colon \alpha < \mathfrak{c}\}$ and consider an analytic function f, say $f = f_\beta$. Fix γ such that $\beta \leqslant \gamma < \mathfrak{c}$. Suppose we had

$$(f_\beta + c_\gamma)(y_0, \ldots, y_{n-1}) = y_n \tag{$*$}$$

where $y_i \in A$ for each $i \leqslant n$. Write $y_i = x_{\alpha_i}$ for each $i \leqslant n$ and let α be larger than γ and larger than any of the indices α_i, $i \leqslant n$. The equality $(*)$ then contradicts (2). Since the set $\{c_\gamma\colon \beta \leqslant \gamma < \mathfrak{c}\}$ meets every uncountable Borel set, we are done. □

Suppose there is a Lusin set $L \subseteq \mathbb{R}^{n+1}$, i.e., a set which is uncountable but has countable intersection with every first category set. (Such a set exists under the Continuum Hypothesis or after uncountably many Cohen reals are added.) By replacing L with the union of its translates by the members of a countable dense set, we can assume that L has uncountable intersection with every ball and hence is everywhere second category. We have that $L \cap (f + c)$ is countable for every continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ and every $c \in \mathbb{R}$ since the graph of $f + c$ is a closed nowhere dense set. (Even if f is merely a Borel function, the Kuratowski–Ulam theorem shows that the graph of f is a set of first category in $\mathbb{R}^{n+1}$ and hence the sections $L \cap (f + c)$ are still countable.)

In spite of all these examples, it is consistent relative to ZFC that second category sets must have large continuous sections. The fundamental result in this direction was proven by Ciesielski and Shelah.

Theorem 1.3. [5, Theorem 2] *If* ZFC *is consistent, then so is* ZFC + *the following statement.*

For every $A \subseteq 2^\omega \times 2^\omega$ for which the sets A and $A^c = (2^\omega \times 2^\omega \setminus A)$ are everywhere second category in $2^\omega \times 2^\omega$ there is a homeomorphism $f\colon 2^\omega \to 2^\omega$ such that the set $\{x \in 2^\omega\colon (x, f(x)) \in A\}$ does not have the property of Baire in 2^ω.

(A set S in a Polish space has the property of Baire if $S \triangle B$ is first category for some Borel set B.)

Shelah proved the following theorem as part of the proof of [12, Theorem 4.7], which states that if ZFC is consistent then so is ZFC + $2^\omega = \omega_2$ + "There is a universal (linear) order of power ω_1."

Theorem 1.4. *If* ZFC *is consistent, then so is* ZFC + *both of the following statements.*

(a) *There is a second category set in $\mathbb{R}$ of cardinality ω_1.*
(b) *Let A and B be everywhere second category subsets of $\mathbb{R}$ of cardinality ω_1. Then A and B are order-isomorphic.*

In light of the similarities between the proofs of Theorems 1.3 and 1.4, it is natural to ask whether the homeomorphism in Theorem 1.3 can be taken to be an order-isomorphism of 2^ω, ordered lexicographically. (Note that an order-isomorphism is necessarily a homeomorphism.) Ciesielski and Natkaniec showed that the proof of Theorem 1.3 can be adapted to give an affirmative answer for $\mathbb{R}$. (See Remark 1.7(1) for the Cantor set.)

Theorem 1.5. [4, Theorem 12A] *If* ZFC *is consistent, then so is* ZFC $+ 2^{\aleph_0} = \aleph_2 +$ *the following statements.*

(a) *Every everywhere second category set in $\mathbb{R}$ has an everywhere second category subset of cardinality ω_1.*
(b) *For every family $\mathcal{A}$ consisting of $\aleph_1$ pairwise disjoint everywhere second category sets in $\mathbb{R}^2$, there is an increasing homeomorphism $f:\mathbb{R}\to\mathbb{R}$ such that $A\cap f$ is everywhere second category in f for every $A\in\mathcal{A}$.*

In [2], we showed that the order-isomorphisms in Theorem 1.4 can be taken to be the restrictions to $\mathbb{R}$ of entire functions and to be arbitrarily good asymptotic approximations (in the sense of the theorems of [8]) of a given non-decreasing surjection of $\mathbb{R}$ and its derivatives. The main result of the present paper is the following theorem, which shows that consistently for any everywhere second category set $A\subseteq\mathbb{R}^{n+1}$, we can find a function $f:\mathbb{R}^n\to\mathbb{R}$ which is the restriction of an entire function $\mathbb{C}^n\to\mathbb{C}$ such that $A\cap f$ is everywhere second category relative to the graph of f. The proof builds on the ideas in [5] and on the argument in [2], among other things extending the techniques from the latter to functions of several variables.

Theorem 1.6. *If* ZFC *is consistent, then so is* ZFC $+ 2^{\aleph_0} = \aleph_2 +$ *the following statements.*

(a) *Every second category set in $\mathbb{R}$ has a second category subset of cardinality $\aleph_1$.*
(b) *For every positive integer n and every collection $\mathcal{C}$ consisting of $\aleph_1$ everywhere second category subsets of $\mathbb{R}^{n+1}$, there is a function $f:\mathbb{R}^n\to\mathbb{R}$ which is the restriction to $\mathbb{R}^n$ of an entire function $\mathbb{C}^n\to\mathbb{C}$ such that $C\cap f$ is everywhere second category in f for every $C\in\mathcal{C}$.*
(c) *Suppose that in* (b) *we are additionally given a positive continuous function $\varepsilon:\mathbb{R}^n\to\mathbb{R}$, a function $g:\mathbb{R}^n\to\mathbb{R}$, a countable dense set $A\subseteq\mathbb{R}^n$, and countable dense sets $B_x\subseteq\mathbb{R}$, $x\in A$.*
   (i) *If k is a nonnegative integer and g is of class C^k, then we may ask that for all multiindices $\alpha=(\alpha_0,\dots,\alpha_{n-1})$ of order at most k and all $x\in\mathbb{R}^n$, $|D^\alpha f(x)-D^\alpha g(x)|<\varepsilon(x)$.*
   (ii) *If $0\leqslant c_0\leqslant c_1\leqslant\cdots$ satisfies $\lim_{i\to\infty}c_i=\infty$ and g is of class C^∞, then we may ask that for every $i<\omega$, for every multiindex α of order at most i and for each $x\in\mathbb{R}^n$ such that $|x|\geqslant c_i$, $|D^\alpha f(x)-D^\alpha g(x)|<\varepsilon(x)$.*
   (iii) *We may ask that for each $x\in A$, $f(x)\in B_x$. Moreover, for any dense $A'\subseteq A$, if the sets B_x, $x\in A'$, are all equal to some $B\subseteq\mathbb{R}$ and for all $x\in A\setminus A'$ we have $B_x\cap B=\emptyset$, then $f[A']$ is an interval of B.*

Remark 1.7. (1) If statements (b) and (c) from Theorem 1.6 hold, then we have the following version of Theorem 1.3. For every collection $\mathcal{C}$ consisting of ω_1 everywhere second category subsets of $2^\omega\times2^\omega$, there is an order-isomorphism $f:2^\omega\to2^\omega$ (for the lexicographic order) such that $C\cap f$ is everywhere second category in f for every $C\in\mathcal{C}$. (*Proof*: Transfer the collection $\mathcal{C}$ to a collection $\mathcal{C}_0$ of subsets of $[0,1]^2$ via the usual dyadic expansion map $2^\omega\to[0,1]$. Via any order-isomorphism of $(0,1)$ with $\mathbb{R}$, transfer $\mathcal{C}_0$ to a collection $\mathcal{C}_1$ of every second category subsets of the plane. Write A for the image of the dyadic rationals in $(0,1)$ under this order-isomorphism. Using (b) and (c), get $f_1:\mathbb{R}\to\mathbb{R}$ such that in particular f_1 and Df_1 are uniformly within $\varepsilon=1/2$ of g and Dg respectively, where g is the identity map. Then f_1 is surjective and $Df_1>0$. Hence f_1 is an order-isomorphism. We may also ask that $f_1(a)\in A$ for each $a\in A$ and $f_1[A]$ is an interval of A. Since f_1 is an order-isomorphism of $\mathbb{R}$, $f_1[A]$ is unbounded and hence $f_1[A]=A$. Pulling this back to $(0,1)$ we get an order-isomorphism f_0 of $(0,1)$ which maps the set of dyadic rationals precisely to itself and meets every member of $\mathcal{C}_0$ in a set which is everywhere second category in f_0. Extend f_0 to $[0,1]$ by letting $f_0(0)=0$, $f_0(1)=1$. Because f_0 maps the set of dyadic rationals to itself, it corresponds in an obvious way to an order-isomorphism f of 2^ω under the usual map, and this f is as desired.)

(2) We cannot allow $|\mathcal{C}|=2^{\aleph_0}$ in (b) even if the members of $\mathcal{C}$ are Borel sets since $\mathcal{C}=\{\mathbb{R}^2\setminus f\colon f$ is a first category Borel set$\}$ would then be a counterexample to the statement even if we weaken the conclusion to say that f is a Borel map instead of a homeomorphism.

(3) The proof of [4, Proposition 13] with minor changes gives the following.

If (a) and (b) hold, then for any everywhere second category set $A\subseteq\mathbb{R}^{n+1}$, there is a function $f:\mathbb{R}^n\to\mathbb{R}$ which is the restriction to $\mathbb{R}^n$ of an entire function and is such that $\{c\in\mathbb{R}\colon A\cap(f+c)$ is everywhere second category in $f+c\}$ is everywhere second category in $\mathbb{R}$. Indeed, the latter set can be taken to include any everywhere second category set B of cardinality $\aleph_1$ specified in advance. (*Proof*: Writing $A - c = \{(x, y - c) \in \mathbb{R}^n \times \mathbb{R}\colon (x, y) \in A\}$, take $\mathcal{C} = \{A - c\colon c \in B\}$.)

(4) By iterated applications of (b), we can get by backwards induction on m, for sets in $\mathbb{R}^{n+1}$, sections having dimension m for any $m = 1, \ldots, n$. For example, if $A \subseteq \mathbb{R}^3$ is everywhere second category and $f : \mathbb{R}^2 \to \mathbb{R}$ is (the restriction to $\mathbb{R}^2$ of) an entire function such that $A \cap f$ is everywhere second category in f, then we can consider the corresponding everywhere second category subset of $\mathbb{R}^2$ obtained via the natural identification of $\mathbb{R}^2$ with the graph of f, i.e., $(x, y) \mapsto (x, y, f(x, y))$. For this set we can get a section determined by an entire function $g : \mathbb{R} \to \mathbb{R}$ so that the section is everywhere second category relative to g. Then we get that A has a relatively second category trace on the curve $x \mapsto f(x, g(x), f(x, g(x)))$.

In this paper, the αth derivative of a function f is denoted exclusively by $D^\alpha f$. Primes do not denote derivatives. We use standard multiindex notation for the mixed partial derivatives of a function $f : \mathbb{C}^n \to \mathbb{C}$ or $f : \mathbb{R}^n \to \mathbb{R}$. If $\alpha = (\alpha_0, \ldots, \alpha_{n-1})$ and $\beta = (\beta_0, \ldots, \beta_{n-1})$ are sequences of nonnegative integers, then we write

$$|\alpha| = \alpha_0 + \cdots + \alpha_{n-1}, \qquad D^\alpha f = \frac{\partial^{\alpha_0 + \cdots + \alpha_{n-1}} f}{\partial^{\alpha_0} z_0 \cdots \partial^{\alpha_{n-1}} z_{n-1}}, \qquad \binom{\alpha}{\beta} = \binom{\alpha_0}{\beta_0} \cdots \binom{\alpha_{n-1}}{\beta_{n-1}}$$

and $\sum_{\beta=0}^{\alpha}$ abbreviates $\sum_{\beta_0=0}^{\alpha_0} \cdots \sum_{\beta_{n-1}=0}^{\alpha_{n-1}}$. Recall that we have the following formulas for numbers x, y and suitably differentiable functions f, g.

$$(x + y)^{|\alpha|} = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} x^{|\beta|} y^{|\alpha| - |\beta|}, \qquad D^\alpha (fg) = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \big(D^\beta f\big)\big(D^{\alpha-\beta} g\big).$$

For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm of x. On $\mathbb{C}$, $|z|$ is the usual modulus and for $z \in \mathbb{C}^n$, we write $|z| = (\sum_{i<n} |z_i|^2)^{1/2}$.

We make use of the following result on asymptotic approximation of differentiable real functions by entire ones. It is a multivariate analog of the strengthening of Carleman's theorem by Hoischen used in [2].

Theorem 1.8. ([8], [6, Theorem and Remark (2)]) *Let $n \in \mathbb{N}$ and $k < \omega$. If $g : \mathbb{R}^n \to \mathbb{R}$ is a function of class C^k and $\varepsilon : \mathbb{R}^n \to \mathbb{R}$ is a positive continuous function, then there exists an entire function $f : \mathbb{C}^n \to \mathbb{C}$ such that $f[\mathbb{R}^n] \subseteq \mathbb{R}$ and for all α such that $|\alpha| \leqslant k$ and all $x \in \mathbb{R}^n$, $|D^\alpha f(x) - D^\alpha g(x)| < \varepsilon(x)$. Furthermore, if g is of class C^∞ and $\{c_i\}_{i<\omega}$ is any non-decreasing sequence of nonnegative real numbers with $\lim c_i = \infty$ then, for every positive continuous ε on $\mathbb{R}^n$ there exists an entire function f such that for all $i < \omega$, all α such that $|\alpha| \leqslant i$ and all $x \in \mathbb{R}^n$ such that $|x| \geqslant c_i$, $|D^\alpha f(x) - D^\alpha g(x)| < \varepsilon(x)$.*

As mentioned earlier, this paper builds on the work in [2]. In a few places there is some duplication of arguments between the two papers. We have chosen to repeat those arguments for the sake of readability and to make this paper self-contained. In Section 2, we define the class of entire functions which will be used in the proof of Theorem 1.6 and establish some of its properties. The class is a several variable analog of the one used in the single variable setting of [2]. The main lemma needed for the proof of Theorem 1.6 is established in Section 3. In the final section of the paper, we indicate how to reduce the main theorem to the setting of the main lemma. The theorem then follows from the main lemma by standard oracle-cc techniques which we sketch.

2. Preliminary results

Until the last section of this paper, we fix a positive integer t. We will be working in $\mathbb{R}^t$ and $\mathbb{C}^t$.

The main goal of the present section is to define a family $H \operatorname{span} \mathcal{G}_0$ of entire functions and prove Proposition 2.4 which shows how members of this family can be approximated in smaller models of set theory and perturbed slightly to alter their values at certain points. The entire functions in the statement of the main theorem will be constructed as limits of sequences of members of this family. The results here are multivariate analogs of results in [2]. We introduce families of entire functions $\mathcal{G}$, $\mathcal{G}_0$ and we show that they have properties which closely resemble those of the corresponding families in [2]. The proof of the main approximation result, Proposition 2.4 can then be established using the corresponding result of [2] as a guide. Some of the details carry over with only minor changes or reinterpretations for functions of several variables. As explained in the introduction, in the interest of readability and making the treatment here self-contained, we provide full proofs.

The function H given by the following proposition serves as an envelope which controls the behavior at infinity of the members of the family $\mathcal{G}_0$ defined below.

Proposition 2.1. *Let $\zeta:\mathbb{R}^t \to \mathbb{R}$ be given by $\zeta(x) = (1+|x|)^{-1}$. For any non-decreasing sequence $\{c_i\}_{i<\omega}$ of non-negative real numbers with $\lim c_i = \infty$ and any positive continuous function $\varepsilon:\mathbb{R}^t \to \mathbb{R}$, there is an entire function $H:\mathbb{C}^t \to \mathbb{C}$ such that*

(a) $H[\mathbb{R}^t] \subseteq \mathbb{R}$,
(b) *for all $x \in \mathbb{R}^t$, $H(x) > 0$,*
(c) *for all $i < \omega$, all multiindices α such that $|\alpha| \leqslant i$ and all $x \in \mathbb{R}^t$ such that $|x| \geqslant c_i$, $|D^\alpha H(x)| < 2^{-i}\zeta(x)\varepsilon(x)$.*

Remark 2.2. (1) All that matters concerning the choice of ζ is that ζ is a continuous function such that $0 < \zeta(x) \leqslant 1$ for all $x \in \mathbb{R}^t$ and $\lim_{|x|\to\infty} \zeta(x) = 0$.

(2) It would be equivalent to state the proposition with $\varepsilon(x)$ instead of $2^{-i}\zeta(x)\varepsilon(x)$ in part (c), but the present formulation is more convenient for our purposes.

Proof. Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be a locally finite partition of unity for $\mathbb{R}^t$ consisting of C^∞ functions whose supports are compact. For a suitable choice of coefficients $\varepsilon_k > 0$, $k \in \mathbb{N}$, the function $h = \sum_{k=1}^\infty \varepsilon_k\varphi_k$ is positive and satisfies that for all $i < \omega$ and all $x \in \mathbb{R}^t$ such that $|x| \geqslant c_i$, $h(x) < 2^{-i}\zeta(x)\varepsilon(x)$. Then some other function H' of the same form is positive and satisfies for all $i < \omega$, all multiindices α such that $|\alpha| \leqslant i$ and all $x \in \mathbb{R}^t$ such that $|x| \geqslant c_i$, $|D^\alpha H'(x)| < \frac{1}{2}h(x)$. By Theorem 1.8, there is an entire function H such that $H[\mathbb{R}^t] \subseteq \mathbb{R}$, $H(x) > 0$ for all $x \in \mathbb{R}$, and for all $i < \omega$, all multiindices α such that $|\alpha| \leqslant i$ and all $x \in \mathbb{R}^t$ such that $|x| \geqslant c_i$, we have $|D^\alpha H(x) - D^\alpha H'(x)| < \frac{1}{2}h(x)$ and hence $|D^\alpha H(x)| \leqslant |D^\alpha H(x) - D^\alpha H'(x)| + |D^\alpha H'(x)| < \frac{1}{2}h(x) + \frac{1}{2}h(x) = h(x) < 2^{-i}\zeta(x)\varepsilon(x)$, giving (c). □

Let $\mathcal{G}$ be the family of entire functions $g(n, A)$ where $n \in \mathbb{N}$, $A \subseteq \mathbb{R}^t$ is a nonvoid finite set, and for all $z \in \mathbb{C}^t$,

$$g(n, A)(z) = \frac{1}{t^{|A|}} \prod_{a\in A} \sum_{k<t} \sin^2\left(\frac{z_k - a_k}{n}\right).$$

These functions were chosen for the property that $g(n, A)$ is zero at the elements of A and for a fixed A and $x \in \mathbb{R}^t \setminus A$, it is easy to pick n so that $g(n, A)(x) \neq 0$. Also important is the property that it is easy to obtain useful bounds on the derivatives of these functions.

Let $\mathcal{G}_0$ be the subfamily consisting of those functions $g(n, A)$ for which $n \geqslant 8|A|$. The next proposition gathers some simple properties of the collection $\mathcal{G}$. It is analogous to [2, Proposition 3.4].

Proposition 2.3. *The family $\mathcal{G}$ has the following properties*:

(a) *For all α and all $x \in \mathbb{R}^t$, $|D^\alpha g(n, A)(x)| \leqslant (2|A|/n)^{|\alpha|}$.*
(b) *Let I_a, $a \in A$, be pairwise disjoint open rectangles in $\mathbb{R}^t$ such that $a \in I_a$, for each $a \in A$. For each $a \in A$, let $(r^{a,m}: m \in \mathbb{N})$ be a sequence in I_a such that for each $m \in \mathbb{N}$ we have $|a - r^{a,m}| < 1/m$. Let $A(m) = \{r^{a,m}: a \in A\}$. Then for all α, $m \in \mathbb{N}$ and $x \in \mathbb{R}^t$,*

$$\left|D^\alpha g(n, A)(x) - D^\alpha g\big(n, A(m)\big)(x)\right| \leqslant m^{-1}\big(2|A|/n\big)^{|\alpha|+1}.$$

and for all $z \in \mathbb{C}^t$, $m \in \mathbb{N}$ and $\lambda, \mu \in \mathbb{R}$, if $M > 0$ and $|z| \leqslant M$ then $|g(n, A(m))(z)| \leqslant T_1$ and

$$\left|\lambda g(n, A)(z) - \mu g\big(n, A(m)\big)(z)\right| \leqslant T_2\big(|\lambda - \mu| + |\mu|/m\big)$$

where T_1 and T_2 are constants which depend only on n, A and M.

(c) *Let $\{c_i\}_{i<\omega}$ be a nondecreasing sequence of nonnegative real numbers with $\lim c_i = \infty$ and let $\varepsilon:\mathbb{R}^t \to \mathbb{R}$ be a positive continuous function. Let $H:\mathbb{C}^t \to \mathbb{C}$ be as given by Proposition 2.1. Write $H \operatorname{span}\mathcal{G}_0$ for the set of functions of the form HG where $G \in \operatorname{span}\mathcal{G}_0$ and $\operatorname{span}\mathcal{G}_0$ is the set of all real linear combinations of elements* *of $\mathcal{G}_0$. Then for all $i < \omega$, for all α such that $|\alpha| \leqslant i$, for all $f \in H \operatorname{span} \mathcal{G}_0$ and for all $x \in \mathbb{R}^t$ such that $|x| \geqslant c_i$, we have $|D^\alpha f(x)| \leqslant (\sum_{g \in \mathcal{G}'} |\lambda_g|)(3/4)^{|\alpha|}\zeta(x)\varepsilon(x)$, where $f = H \sum\{\lambda_g g\colon\ g \in \mathcal{G}'\}$ for some finite $\mathcal{G}' \subseteq \mathcal{G}_0$ and $\lambda_g \in \mathbb{R}$, $g \in \mathcal{G}'$. (ζ is as in Proposition 2.1.)*

Proof. (a) Note that the derivative of the function $\bar{g} : \mathbb{C} \to \mathbb{C}$ defined by $\bar{g}(z) = \sin^2((z-a)/n)$ is given by

$$D\bar{g}(z) = \frac{2}{n}\sin\left(\frac{z-a}{n}\right)\cos\left(\frac{z-a}{n}\right) = \frac{1}{n}\sin\left(\frac{2z-2a}{n}\right)$$

and hence the jth derivative, for $j \geqslant 1$, has the form

$$D^j\bar{g}(z) = \pm\frac{2^{j-1}}{n^j} f\left(\frac{2z-2a}{n}\right),$$

where f is either sine or cosine. By induction on i, it follows that the ith order derivatives of $g(n, A)$ can be expressed as a sum of $|A|^i$ terms each of which has the form $\pm 2^j/n^i$, for some $j \leqslant \max(0, i-1)$, times $t^{-|A|}$ times a product indexed by A in which the factor corresponding to $a \in A$ either is zero or is $\sum_{k<t}\sin^2((z_k - a_k)/n)$ or has the form $f((2z_k - 2a_k)/n)$, where f is sine or cosine and $k < t$. For $z \in \mathbb{R}$, the sines and cosines are bounded in absolute value by 1 and hence these products indexed by A are bounded by $t^{|A|}$. We get that for $x \in \mathbb{R}^t$ and $|\alpha| = i$, $|D^\alpha g(n, A)(x)| \leqslant (2|A|/n)^i$ and hence (a) holds.

(b) We use the simple estimate

$$|u_1u_2\cdots u_k - v_1v_2\cdots v_k| \leqslant \sum_{i=1}^{k}\mu_0^{k-1}|u_i - v_i| \tag{$*$}$$

which holds for any natural number k and for any numbers $u_i, v_i \in \mathbb{C}$ of modulus at most μ_0, $i = 1, \ldots, k$. For $x \in \mathbb{R}$, letting $|\alpha| = i$, the difference between $D^\alpha g(n, A)(x)$ and $D^\alpha g(n, A(m))(x)$ can be expressed, as in the argument for (a), as a sum of $|A|^i$ terms each of which has the form $\pm 2^j/n^i$, for some $j \leqslant \max(0, i-1)$, times $t^{-|A|}$ times a difference of the form

$$\prod_{a\in A} f_a(x, a, n) - \prod_{a\in A} f_a\big(x, r^{a,m}, n\big),$$

where, for each $a \in A$, $f_a(x, y, n) = 0$ or $f_a(x, y, n) = \sum_{k<t}\sin^2((x_k - y_k)/n)$ or $f_a(x, y, n) = \sin((2x_k - 2y_k)/n)$ or $f_a(x, y, n) = \cos((2x_k - 2y_k)/n)$. We now apply $(*)$ with $\mu_0 = t$ to these differences. Note that, by the Mean Value Theorem, for $x \in \mathbb{R}$ and $a \in A$ we have

(1) when $f_a(x, y, n) = \sum_{k<t}\sin^2((x_k - y_k)/n)$:

$$\begin{aligned}\big|f_a(x, a, n) - f_a\big(x, r^{a,m}, n\big)\big| &\leqslant \sum_{k<t}\big|\sin^2\big((x_k - a_k)/n\big) - \sin^2\big(\big(x_k - r_k^{a,m}\big)/n\big)\big| \\ &\leqslant \sum_{k<t}(1/n)\big|r_k^{a,m} - a_k\big| \leqslant (t/n)\big|r^{a,m} - a\big| < t/(mn),\end{aligned}$$

(2) when $f_a(x, y, n) = \sin((2x_k - 2y_k)/n)$:

$$\big|f_a(x, a, n) - f_a\big(x, r^{a,m}, n\big)\big| \leqslant (2/n)\big|r_k^{a,m} - a_k\big| \leqslant (2/n)\big|r^{a,m} - a\big| < 2/(mn)$$

and similarly when $f_a(x, a, n) = \cos((2x_k - 2a_k)/n)$. This gives that for each $x \in \mathbb{R}^t$,

$$\big|D^\alpha g(n, A)(x) - D^\alpha g\big(n, A(m)\big)(x)\big| \leqslant \frac{2^{i-1}|A|^i}{n^i}t^{-|A|}\sum_{a\in A}t^{|A|-1}\frac{\max(2, t)}{mn} \leqslant \frac{2^i|A|^{i+1}}{mn^{i+1}}.$$

(The 2^{i-1} should be $2^i = 1$ when $i = 0$, but then the $\max(2, t)$ bound can be replaced by t (because case (2) is irrelevant) so the upper bound obtained here is valid.) This takes care of the first part of (b).

For the second, we have that if $|z| \leqslant M$, then for each $a \in A$ and $k < t$,

$$\big|z_k - r_k^{a,m}\big| \leqslant |z_k - a_k| + \big|a_k - r_k^{a,m}\big| \leqslant M + |a| + 1.$$

Hence, letting T_1 denote the supremum of

$$t^{-|A|}\prod_{a\in A}\sum_{k<t}\bigl|\sin^2\bigl(z_k^a/n\bigr)\bigr|$$

over all choices of $z^a\in\mathbb{C}^t$ such that for all $k<t$ and $a\in A$, $|z_k^a|\leqslant M+|a|+1$, we get $|g(n,A(m))(z)|\leqslant T_1$. For the remaining part of (b), use

$$\begin{aligned}
&\sin^2\left(\frac{z_k-a_k}{n}\right)-\sin^2\left(\frac{z_k-r_k^{a,m}}{n}\right)\\
&\quad=\left(\sin\left(\frac{z_k-a_k}{n}\right)+\sin\left(\frac{z_k-r_k^{a,m}}{n}\right)\right)\left(\sin\left(\frac{z_k-a_k}{n}\right)-\sin\left(\frac{z_k-r_k^{a,m}}{n}\right)\right)\\
&\quad=2\left(\sin\left(\frac{z_k-a_k}{n}\right)+\sin\left(\frac{z_k-r_k^{a,m}}{n}\right)\right)\cos\left(\frac{2z_k-r_k^{a,m}-a_k}{2n}\right)\sin\left(\frac{r_k^{a,m}-a_k}{2n}\right)
\end{aligned}$$

and apply (∗) to the difference $g(n,A)(z)-g(n,A(m))(z)$ with μ_0 taken to be the supremum of 1 and all the quantities

(i) $|\sin^2((z'-a')/n)|$, $|z'|\leqslant M$, $a'\in\{a_k\colon a\in A,\ k<t\}$,
(ii) $|2(\sin((z'-a')/n)+\sin((z'-r')/n))\cos((2z'-r'-a')/(2n))|$, $|z'|\leqslant M$, $a'\in\{a_k\colon a\in A,\ k<t\}$, $r'\in\bigcup_{a\in A,k<t}[a_k-1,a_k+1]$,
(iii) $|g(n,A)(z)|$, $|z|\leqslant M$.

For $|z|\leqslant M$ we have

$$\begin{aligned}
\bigl|g(n,A)(z)-g\bigl(n,A(m)\bigr)(z)\bigr|&=\frac{1}{t^{|A|}}\left|\prod_{a\in A}\sum_{k<t}\sin^2\left(\frac{z_k-a_k}{n}\right)-\prod_{a\in A}\sum_{k<t}\sin^2\left(\frac{z_k-r_k^{a,m}}{n}\right)\right|\\
&\leqslant\frac{1}{t^{|A|}}\sum_{a\in A}(t\mu_0)^{|A|-1}\sum_{k<t}\left|\sin^2\left(\frac{z_k-a_k}{n}\right)-\sin^2\left(\frac{z_k-r_k^{a,m}}{n}\right)\right|\\
&\leqslant\frac{1}{t}\sum_{a\in A}\mu_0^{|A|}\sum_{k<t}\left|\sin\left(\frac{r_k^{a,m}-a_k}{2n}\right)\right|.
\end{aligned}$$

Since $|\sin((r_k^{a,m}-a_k)/(2n))|\leqslant|(r_k^{a,m}-a_k)/(2n)|\leqslant 1/m$, we have

$$\begin{aligned}
\bigl|\lambda g(n,A)(z)-\mu g\bigl(n,A(m)\bigr)(z)\bigr|&\leqslant|\lambda-\mu|\bigl|g(n,A)(z)\bigr|+|\mu|\bigl|g(n,A)(z)-g\bigl(n,A(m)\bigr)(z)\bigr|\\
&\leqslant\mu_0|\lambda-\mu|+|A|\mu_0^{|A|}|\mu|/m\\
&\leqslant|A|\mu_0^{|A|}\bigl(|\lambda-\mu|+|\mu|/m\bigr).
\end{aligned}$$

(c) Let $f\in H\operatorname{span}\mathcal{G}_0$ and let α be a multiindex. Write

$$f=H\sum\{\lambda_s s\colon s\in\mathcal{G}'\}$$

for some finite $\mathcal{G}'\subseteq\mathcal{G}_0$ and $\lambda_s\in\mathbb{R}$, $s=g(n_s,A_s)\in\mathcal{G}'$. We have

$$D^\alpha f=\sum\bigl\{\lambda_s D^\alpha(Hs)\colon s\in\mathcal{G}'\bigr\}.$$

For each $s\in\mathcal{G}'$ we have, using (a), the bound (for $x\in\mathbb{R}$)

$$\bigl|D^\alpha(Hs)(x)\bigr|\leqslant\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\bigl|D^\beta H(x)\bigr|\bigl|D^{\alpha-\beta}s(x)\bigr|\leqslant\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\bigl|D^\beta H(x)\bigr|\bigl(2|A_s|/n_s\bigr)^{|\alpha|-|\beta|}.$$

Using $n_s\geqslant 8|A_s|$, this gives

$$\bigl|D^\alpha f(x)\bigr|\leqslant\sum_{s\in\mathcal{G}'}|\lambda_s|\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\bigl|D^\beta H(x)\bigr|\frac{1}{4^{|\alpha|-|\beta|}}.$$

When $|\alpha| \leqslant i$ and $|x| \geqslant c_i$, we have $|x| \geqslant c_k$ for $k = 0, \dots, i$ and so

$$\left|D^\alpha f(x)\right| \leqslant \sum_{s \in \mathcal{G}'} |\lambda_s| \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \frac{1}{2^{|\beta|}} \frac{1}{4^{|\alpha|-|\beta|}} \zeta(x)\varepsilon(x) = \left(\sum_{s \in \mathcal{G}'} |\lambda_s|\right)(3/4)^{|\alpha|}\zeta(x)\varepsilon(x). \qquad \square$$

Proposition 2.4. *Let N be an elementary submodel of H_θ for some regular $\theta > \mathfrak{c}$. Let $\{c_i\}_{i<\omega} \in N$ be a nondecreasing sequence of nonnegative real numbers with $\lim c_i = \infty$. Let $\varepsilon : \mathbb{R}^t \to \mathbb{R}$ be positive and continuous, $\varepsilon \in N$. Let $H : \mathbb{C}^t \to \mathbb{C}$ be as given by Proposition 2.1, $H \in N$. Let $M > 0$. Let B be a countable dense subset of $\mathbb{R}^t$ and let C be countable dense subsets of $\mathbb{R}$ with $B, C \in N$. Let $f \in H \operatorname{span} \mathcal{G}_0$. Let $K_0, K_1 \subseteq \mathbb{R}^t$ be finite such that $K_0, f[K_1] \in N$. Let $h, h' \subseteq f \restriction \mathbb{R}^t$ be finite such that $h \in N$. Assume that the sets $\operatorname{dom} h$, $\operatorname{dom} h'$, K_0, K_1 are pairwise disjoint. For each $u \in \operatorname{dom} h'$, let $S_u \subseteq \mathbb{R}^t$ be a set whose closure contains u. Then there are a function $f' \in N \cap (H \operatorname{span} \mathcal{G}_0)$, distinct points $b_x \in \mathbb{R}^t$, $x \in K_1$, and distinct points $d_u \in \mathbb{R}^t$, $u \in \operatorname{dom} h'$, such that the sets K_0, $\{b_x\colon x \in K_1\}$, $\operatorname{dom} h$, $\{d_u\colon u \in \operatorname{dom} h'\}$ are pairwise disjoint and for some $n \in \mathbb{N}$ and positive rational number r, the following properties hold. In this list, K denotes the set $K_0 \cup \{b_x\colon x \in K_1\} \cup \operatorname{dom} h \cup \{d_u\colon u \in \operatorname{dom} h'\}$:*

(a) $h \subseteq f'$;
(b) *for all $x \in K_0$, $f'(x) \in C$;*
(c) *for all $x \in K_1$, $b_x \in B$, $|b_x - x| < \varepsilon(0)$ and $f'(b_x) = f(x)$;*
(d) *for all $u \in \operatorname{dom} h'$, $d_u \in S_u$ and $|d_u - u| < \varepsilon(0)$;*
(e) *$n \geqslant 8|K|$ and for all $i < \omega$, for all α such that $|\alpha| \leqslant i$ and for all $\sigma_u \in \mathbb{R}$ such that $|\sigma_u| \leqslant r$ for each $u \in \operatorname{dom} h'$, we have that for $x \in \mathbb{R}^t$ such that $|x| \geqslant c_i$,*

$$\left|D^\alpha f(x) - D^\alpha\left(f' + \sum_{u \in \operatorname{dom} h'} \sigma_u H g\bigl(n, K \setminus \{d_u\}\bigr)\right)(x)\right| < \varepsilon(x);$$

(f) *the determinant of the $|\operatorname{dom} h'| \times |\operatorname{dom} h'|$ matrix*

$$\bigl(g\bigl(n, K \setminus \{d_u\}\bigr)(v)\bigr)_{u,v \in \operatorname{dom} h'}$$

is not zero and for some real numbers σ_u such that $|\sigma_u| < r$, $u \in \operatorname{dom} h'$, we have

$$h' \subseteq f' + \sum_{u \in \operatorname{dom} h'} \sigma_u H g\bigl(n, K \setminus \{d_u\}\bigr);$$

(g) *for all $z \in \mathbb{C}^t$ such that $|z| \leqslant M$ and for all $\sigma_u \in \mathbb{R}$ such that $|\sigma_u| \leqslant r$, $u \in \operatorname{dom} h'$,*

$$r \sum_{u \in \operatorname{dom} h'} \left|H(z) g\bigl(n, K \setminus \{d_u\}\bigr)(z)\right| < \varepsilon(0)$$

and

$$\left|f(z) - \left(f' + \sum_{u \in \operatorname{dom} h'} \sigma_u H g\bigl(n, K \setminus \{d_u\}\bigr)\right)(z)\right| < \varepsilon(0).$$

Remark 2.5. If λ is a rational number such that $0 < \lambda < 1$, then we can write $\lambda\varepsilon$ instead of ε in the conclusion. To see that this modified conclusion holds, we argue as follows. First notice that it follows from properties (a), (b) and (c) of Proposition 2.1 that the same properties hold for $(\lambda\varepsilon, \lambda H)$ instead of (ε, H). Since λ is rational, $\lambda\varepsilon, \lambda H \in N$. Since $(\lambda H) \operatorname{span} \mathcal{G}_0 = H \operatorname{span} \mathcal{G}_0$, we have $f \in (\lambda H) \operatorname{span} \mathcal{G}_0$. The proposition therefore gives $f' \in (\lambda H) \operatorname{span} \mathcal{G}_0 = H \operatorname{span} \mathcal{G}_0$, $f' \in N$, $n \in \mathbb{N}$ and a positive rational number r such that (a)–(f) hold with $\lambda\varepsilon$ and λH in the place of ε and H, respectively. But now notice that everywhere H is mentioned, i.e., in (e), (f) and (g), it is multiplied by r or by some σ such that $|\sigma| \leqslant r$. Equivalent statements of these clauses are obtained by restoring λH to H and replacing r by λr, giving the desired modification of the conclusion.

Proof. Choose pairwise disjoint open rectangles in $\mathbb{R}^t$, I_x for $x \in K_1$ and J_u for $u \in \operatorname{dom} h'$ such that for all $x \in K_1$ and $u \in \operatorname{dom} h'$ we have $x \in I_x$, $u \in J_u$, and

$$I_x \cap (K_0 \cup \operatorname{dom} h) = J_u \cap (K_0 \cup \operatorname{dom} h) = \emptyset.$$

For each $x \in K_1$, choose a sequence of points $b^{x,m} \in B \cap I_x$, $m \in \mathbb{N}$, such that $|b^{x,m} - x| < 1/m$. For each $u \in \operatorname{dom} h'$, choose a sequence of points $d^{u,m} \in S_u \cap J_u$, $m \in \mathbb{N}$, such that $|d^{u,m} - u| < 1/m$. Let

$$\overline{K} = K_0 \cup K_1 \cup \operatorname{dom} h,$$
$$\overline{K}_m = K_0 \cup \{b^{x,m}\colon x \in K_1\} \cup \operatorname{dom} h.$$

For each $x \in K_0$, choose a sequence of points $c_{x,m} \in C$, $m \in \mathbb{N}$, such that $|c_{x,m} - f(x)| < 1/m$. Let

$$f = H \sum \{\lambda_s s\colon s \in \mathcal{G}'\}$$

where $\mathcal{G}'$ is a finite subset of $\mathcal{G}_0$ and $\lambda_s \in \mathbb{R}$ for each $s \in \mathcal{G}'$. For each $s \in \mathcal{G}'$, let $n(s) \in \mathbb{N}$ and $A(s) \subseteq \mathbb{R}^t$ be such that $s = g(n(s), A(s))$. For each $s \in \mathcal{G}'$, pick pairwise disjoint open rectangles I^s_a such that $a \in I^s_a$, for $a \in A(s)$. For each $a \in A(s)$, choose a sequence of points with rational coordinates $r^{s,a,m} \in I^s_a$, $m \in \mathbb{N}$, such that for each $m \in \mathbb{N}$ we have $|a - r^{s,a,m}| < 1/m$. Let $A(s,m) = \{r^{s,a,m}\colon a \in A(s)\}$. Choose $n \in \mathbb{N}$ large enough so that

(i) for each $u \in \operatorname{dom} h'$, u is not a zero of $g(n, \overline{K} \cup (\operatorname{dom} h') \setminus \{u\})$ or any $g(n, \overline{K}_m \cup (\operatorname{dom} h') \setminus \{u\})$, $m \in \mathbb{N}$,
(ii) for each $a \in \overline{K}_m$, a is not a zero of $g(n, \overline{K}_m \setminus \{a\})$,
(iii) for each $x \in \overline{K}$, x is not a zero of $g(n, \overline{K} \setminus \{x\})$,
(iv) $n \geqslant 8(|K_0| + |K_1| + |\operatorname{dom} h| + |\operatorname{dom} h'|)$.

(This is possible because the sets $\{b^{x,m}\colon m \in \mathbb{N}\}$ are bounded, so that for large enough n we have that the quantities

$$\left|a_k - b^{x,m}_k\right|,\ \left|u_k - b^{x,m}_k\right| \quad (a \in \overline{K}_m,\ x \in K_1,\ u \in \operatorname{dom} h',\ k < t,\ m \in \mathbb{N})$$

are less than $n\pi$.)

Note that, by (i), the matrix

$$M = \big(g\big(n, \overline{K} \cup (\operatorname{dom} h') \setminus \{u\}\big)(v)\big)_{u,v \in \operatorname{dom} h'}$$

is a diagonal matrix with nonzero entries on the diagonal. In particular, it has a nonzero determinant. Thus, by continuity of the determinant and the fact that for every v

$$\left|g\big(n, \overline{K}_m \cup \{d^{w,m}\colon w \in \operatorname{dom} h',\ w \neq u\}\big)(v) - g\big(n, \overline{K} \cup (\operatorname{dom} h') \setminus \{u\}\big)(v)\right| \leqslant 1/m$$

(which follows from (iv) and Proposition 2.3(b)), we have that for all large enough values of m, the matrix

$$M(m) = \big(g\big(n, \overline{K}_m \cup \{d^{w,m}\colon w \in \operatorname{dom} h',\ w \neq u\}\big)(v)\big)_{u,v \in \operatorname{dom} h'}$$

also has a nonzero determinant.

Consider functions f', $\bar{f}$ of the form

$$f' = H \sum \big\{\mu_s g\big(n(s), A(s,m)\big)\colon s \in \mathcal{G}'\big\} + H \sum \big\{\sigma_a g\big(n, \overline{K}_m \setminus \{a\}\big)\colon a \in \overline{K}_m\big\}$$

and

$$\begin{aligned}\bar{f} &= H \sum \big\{\mu_s g\big(n(s), A(s,m)\big)\colon s \in \mathcal{G}'\big\} + H \sum \big\{\sigma_a g\big(n, \overline{K}_m \setminus \{a\}\big)\colon a \in \overline{K}_m\big\} \\ &\quad + H \sum \big\{\sigma_u g\big(n, \overline{K}_m \cup \{d^{v,m}\colon v \in \operatorname{dom} h',\ v \neq u\}\big)\colon u \in \operatorname{dom} h'\big\} \\ &= f' + H \sum \big\{\sigma_u g\big(n, \overline{K}_m \cup \{d^{v,m}\colon v \in \operatorname{dom} h',\ v \neq u\}\big)\colon u \in \operatorname{dom} h'\big\}\end{aligned}$$

where $m \in \mathbb{N}$, $\mu_s, \sigma_a, \sigma_u \in \mathbb{R}$ ($s \in \mathcal{G}'$, $a \in \overline{K}_m$, $u \in \operatorname{dom} h'$). For $a \in \overline{K}_m$, among the functions $g(n, \overline{K}_m \setminus \{a'\})$, $a' \in \overline{K}_m$, only the one with $a' = a$ is not zero at a. This leads to the following observations.

(1) For $a \in \operatorname{dom} h$ and for each choice of $m \in \mathbb{N}$ and $\vec{\mu} = (\mu_s)_{s \in \mathcal{G}'}$, there is a unique $\sigma_a = \sigma_a(m, \vec{\mu})$ for which $f'(a) = f(a) = h(a)$, namely

$$\sigma_a(m, \vec{\mu}) = \frac{\sum\{\lambda_s g(n(s), A(s))(a) - \mu_s g(n(s), A(s,m))(a)\colon s \in \mathcal{G}'\}}{g(n, \overline{K}_m \setminus \{a\})(a)}.$$

(2) For each $x \in K_0$ and each choice of $m \in \mathbb{N}$ and $\vec{\mu} = (\mu_s)_{s\in\mathcal{G}'}$, there is a unique $\sigma_x = \sigma_x(m, \vec{\mu})$ for which $f'(x) = c_{x,m}$, namely

$$\sigma_x(m,\vec{\mu}) = \frac{c_{x,m} - H(x)\sum\{\mu_s g(n(s), A(s,m))(x)\colon\ s \in \mathcal{G}'\}}{H(x)g(n, \overline{K}_m \setminus \{x\})(x)} = \frac{c_{x,m} - f(x) + H(x)\sum\{\lambda_s g(n(s), A(s))(x) - \mu_s g(n(s), A(s,m))(x)\colon\ s \in \mathcal{G}'\}}{H(x)g(n, \overline{K}_m \setminus \{x\})(x)}.$$

(3) For each $x \in K_1$ and each choice of $m \in \mathbb{N}$ and $\vec{\mu} = (\mu_s)_{s\in\mathcal{G}'}$, there is a unique $\sigma_{b^{x,m}} = \sigma_{b^{x,m}}(m, \vec{\mu})$ for which $f'(b^{x,m}) = f(x)$, namely

$$\begin{aligned}
\sigma_{b^{x,m}}(m,\vec{\mu}) &= \frac{f(x) - H(b^{x,m})\sum\{\mu_s g(n(s), A(s,m))(b^{x,m})\colon\ s \in \mathcal{G}'\}}{H(b^{x,m})g(n, \overline{K}_m \setminus \{b^{x,m}\})(b^{x,m})} \\
&= \frac{H(x)\sum_{s\in\mathcal{G}'}\lambda_s g(n(s), A(s))(x) - H(b^{x,m})\sum_{s\in\mathcal{G}'}\mu_s g(n(s), A(s,m))(b^{x,m})}{H(b^{x,m})g(n, \overline{K}_m \setminus \{b^{x,m}\})(b^{x,m})} \\
&= \Big\{\big[H(x) - H\big(b^{x,m}\big)\big]\sum_{s\in\mathcal{G}'}\lambda_s g\big(n(s), A(s)\big)(x) \\
&\quad + H\big(b^{x,m}\big)\sum_{s\in\mathcal{G}'}\lambda_s\big(g\big(n(s), A(s)\big)(x) - g\big(n(s), A(s)\big)\big(b^{x,m}\big)\big) \\
&\quad + H\big(b^{x,m}\big)\sum_{s\in\mathcal{G}'}\big(\lambda_s g\big(n(s), A(s)\big)\big(b^{x,m}\big) - \mu_s g\big(n(s), A(s,m)\big)\big(b^{x,m}\big)\big)\Big\} \\
&\quad \times \big[H\big(b^{x,m}\big)g\big(n, \overline{K}_m \setminus \big\{b^{x,m}\big\}\big)\big(b^{x,m}\big)\big]^{-1}.
\end{aligned}$$

(4) If m is large enough so that $\det M(m) \neq 0$, then given the assignment of values $\sigma_a = \sigma_a(m, \vec{\mu})$, $a \in \overline{K}_m$, there is a unique value of the vector $(\sigma_v)_{v\in\operatorname{dom} h'} = (\sigma_v(m,\vec{\mu}))_{v\in\operatorname{dom} h'}$ for which $\bar{f}(u) = f(u) = h'(u)$ for each $u \in \operatorname{dom} h'$, namely the solution to the equations

$$\begin{aligned}
&\sum_{v\in\operatorname{dom} h'}\sigma_v(m,\vec{\mu})g\big(n, \overline{K}_m \cup \big\{d^{w,m}\colon\ w \in \operatorname{dom} h',\ w \neq v\big\}\big)(u) \\
&\quad = \sum_{s\in\mathcal{G}'}\big[\lambda_s g\big(n(s), A(s)\big)(u) - \mu_s g\big(n(s), A(s,m)\big)(u)\big] - \sum_{a\in\overline{K}_m}\sigma_a(m,\vec{\mu})g\big(n, \overline{K}_m \setminus \{a\}\big)(u), \\
&\quad u \in \operatorname{dom} h'.
\end{aligned}$$

Note that for functions f', if $\sigma_a = \sigma_a(m, \vec{\mu})$ for each $a \in \overline{K}_m$, then (a), (b) hold and (c), (d) hold (with $b_x = b^{x,m}$, $d_u = d^{u,m}$) if m is large enough. Also, as $\vec{\mu} = (\mu_s)_{s\in\mathcal{G}'} \to (\lambda_s)_{s\in\mathcal{G}'}$ and $m \to \infty$, we have $\sigma_a(m,\vec{\mu}) \to 0$ for each $a \in K_0 \cup \operatorname{dom} h$, $\sigma_{b^{x,m}}(m,\vec{\mu}) \to 0$ for each $x \in K_1$. It follows that $\sigma_u(m,\vec{\mu}) \to 0$ for each $u \in \operatorname{dom} h'$ as well, because by continuity of the entries in the inverse of a matrix as functions of the entries of the original matrix, the inverse $M(m)^{-1}$ converges as $m \to \infty$ (to M^{-1}). The detailed verification, most of which we leave to the reader, uses the consequence of Proposition 2.3 (part (a) and the first part of (b), both with $|\alpha| = 0$) that whenever $n' \geqslant 2|A'|$, we have for all $x \in \mathbb{R}$, $|g(n', A')(x)| \leqslant 1$ and $|g(n', A'(m))(x) - g(n', A')(x)| \leqslant 1/m$. In the present circumstances, we have $\overline{K}$ and $\overline{K}_m$ (as well as the pairs $(\overline{K} \setminus \{a\}, \overline{K}_m \setminus \{a\})$, $a \in K_0 \cup \operatorname{dom} h$, $(\overline{K} \setminus \{x\}, \overline{K}_m \setminus \{b^{x,m}\})$, $x \in K_1$, and $(\overline{K} \cup (\operatorname{dom} h') \setminus \{u\}, \overline{K}_m \cup (\operatorname{dom} h') \setminus \{u\})$, $u \in \operatorname{dom} h'$) playing the role of A' and $A'(m)$. Note in particular that the denominators in (1)–(4) are bounded away from zero. For example, for the denominator in (3) we have that $H(b^{x,m}) \to H(x) > 0$ and, by the choice of n, $g(n, \overline{K} \setminus \{x\})(x) \neq 0$ and

$$\begin{aligned}
&\big|g\big(n, \overline{K}_m \setminus \big\{b^{x,m}\big\}\big)\big(b^{x,m}\big) - g\big(n, \overline{K} \setminus \{x\}\big)(x)\big| \\
&\quad \leqslant \big|g\big(n, \overline{K}_m \setminus \big\{b^{x,m}\big\}\big)\big(b^{x,m}\big) - g\big(n, \overline{K} \setminus \{x\}\big)\big(b^{x,m}\big)\big| + \big|g\big(n, \overline{K} \setminus \{x\}\big)\big(b^{x,m}\big) - g\big(n, \overline{K} \setminus \{x\}\big)(x)\big| \\
&\quad \leqslant 1/m + \big|g\big(n, \overline{K} \setminus \{x\}\big)\big(b^{x,m}\big) - g\big(n, \overline{K} \setminus \{x\}\big)(x)\big| \to 0 \quad \text{as } m \to \infty.
\end{aligned}$$

Fix m_0 such that for $m \geqslant m_0$, $M(m)$ has nonzero determinant. For the rest of the proof, we restrict our attention to choices of the coefficients in the definitions of f' and $\bar{f}$ such that $|\mu_s - \lambda_s| \leqslant 1$, $s \in \mathcal{G}'$, $|\sigma_a| \leqslant 1$, $a \in \overline{K}_m$, and $|\sigma_u| \leqslant 1$, $u \in \operatorname{dom} h'$. Given these restrictions, let $i_0 < \omega$ be such that if $i_0 < |\alpha| \leqslant i$ then for all $m \geqslant m_0$, for all $\vec{\mu} = (\mu_s)_{s \in \mathcal{G}'}$ and for all $x \in \mathbb{R}^t$, we have that if $|x| \geqslant c_i$ then $|D^\alpha(\bar{f} - f)(x)| < \varepsilon(x)$. (Such an i_0 exists by Proposition 2.3(c) which, because $\zeta(x) \leqslant 1$, yields $|D^\alpha(\bar{f} - f)(x)| < C(3/4)^{|\alpha|}\varepsilon(x)$ whenever $|\alpha| \leqslant i$ and $|x| \geqslant c_i$, where C is a constant independent of the choice of the coefficients in the definition $\bar{f}$ as long as the restrictions we just placed on these coefficients are respected.)

Now let L be a closed ball in $\mathbb{R}^t$ such that for all $m \geqslant m_0$, for all $\vec{\mu} = (\mu_s)_{s \in \mathcal{G}'}$ and for all $x \in \mathbb{R}^t \setminus L$ (and given the above restrictions on the choice of coefficients for $\bar{f}$), we have that if $|x| \geqslant c_i$ and $|\alpha| \leqslant i$ then $|D^\alpha(\bar{f} - f)(x)| < \varepsilon(x)$. (Such an L exists by Proposition 2.3(c) again. Whenever $|\alpha| \leqslant i$ and $|x| \geqslant c_i$, $|D^\alpha(\bar{f} - f)(x)| < C(3/4)^{|\alpha|}\zeta(x)\varepsilon(x) \leqslant C\zeta(x)\varepsilon(x) < \varepsilon(x)$ for $|x|$ large enough since $\lim_{|x| \to \infty} \zeta(x) = 0$.)

Henceforth, we limit ourselves to functions f', $\bar{f}$ for which, in addition to the restrictions imposed above, we have $\sigma_a = \sigma_a(m, \vec{\mu})$, $a \in \overline{K}_m$. (So we now consider only coefficients so that $|\sigma_u| \leqslant 1$, $u \in \operatorname{dom} h'$, and for $a \in \overline{K}_m$, $\sigma_a = \sigma_a(m, \vec{\mu})$ with $\vec{\mu}$ close enough to $\vec{\lambda}$ and $m \geqslant m_0$ large enough so that the conditions $|\mu_s - \lambda_s| \leqslant 1$, $s \in \mathcal{G}'$ and $|\sigma_a| \leqslant 1$, $a \in \overline{K}_m$ are satisfied.) Consider the following facts.

(5) if $|\alpha| \leqslant i$ then for $|x| \geqslant c_i$, $|D^\alpha f(x) - D^\alpha \bar{f}(x)| < \varepsilon(x)$ can only fail if $|\alpha| \leqslant i_0$ and $x \in L$,
(6) for some $\delta > 0$, we have $\varepsilon(x) > \delta$ for all $x \in L$,
(7) for $|z| \leqslant M$, we have

$$\begin{aligned}
\bigl|f(z) - \bar{f}(z)\bigr| &\leqslant \bigl|H(z)\bigr| \Bigl[\sum \bigl\{\bigl|\lambda_s g\bigl(n(s), A(s)\bigr)(z) - \mu_s g\bigl(n(s), A(s, m)\bigr)(z)\bigr|\colon s \in \mathcal{G}'\bigr\} \\
&\quad + \sum \bigl\{\bigl|\sigma_a(m, \vec{\mu})\bigr|\bigl|g\bigl(n, \overline{K}_m \setminus \{a\}\bigr)(z)\bigr|\colon a \in \overline{K}_m\bigr\} \\
&\quad + \sum \bigl\{|\sigma_u|\bigl|g\bigl(n, \overline{K}_m \cup \bigl\{d^{v,m}\colon v \in \operatorname{dom} h',\ v \neq u\bigr\}\bigr)(z)\bigr|\colon u \in \operatorname{dom} h'\bigr\}\Bigr] \\
&\leqslant \Bigl(\sup_{|z| \leqslant M} \bigl|H(z)\bigr|\Bigr) \Bigl[\sum \bigl\{T_{s,M}\bigl(|\lambda_s - \mu_s| + |\mu_s|/m\bigr)\colon s \in \mathcal{G}'\bigr\} \\
&\quad + \sum \bigl\{\bigl|\sigma_a(m, \vec{\mu})\bigr| T_{n,M}\colon a \in \overline{K}_m\bigr\} + \sum \bigl\{|\sigma_u| T_{n,M}\colon u \in \operatorname{dom} h'\bigr\}\Bigr],
\end{aligned}$$

where $T_{n,M}$, $T_{s,M}$ are constants independent of m as in Proposition 2.3(b), and
(8) for $|\alpha| \leqslant i_0$ and $x \in L$, we have

$$\begin{aligned}
\bigl|D^\alpha f(x) - D^\alpha \bar{f}(x)\bigr| &\leqslant \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \bigl|D^{\alpha-\beta} H(x)\bigr| \\
&\quad \Bigl[\sum_{s \in \mathcal{G}'} \bigl|\lambda_s D^\beta g\bigl(n(s), A(s)\bigr)(x) - \mu_s D^\beta g\bigl(n(s), A(s, m)\bigr)(x)\bigr| \\
&\quad + \sum_{a \in \overline{K}_m} \bigl|\sigma_a(m, \vec{\mu})\bigr|\bigl|D^\beta g\bigl(n, \overline{K}_m \setminus \{a\}\bigr)(x)\bigr| \\
&\quad + \sum \bigl\{|\sigma_u|\bigl|D^\beta g\bigl(n, \overline{K}_m \cup \bigl\{d^{v,m}\colon v \in \operatorname{dom} h',\ v \neq u\bigr\}\bigr)(x)\bigr|\colon u \in \operatorname{dom} h'\bigr\}\Bigr] \\
&\leqslant \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \bigl|D^{\alpha-\beta} H(x)\bigr| \Bigl[\sum_{s \in \mathcal{G}'} \bigl(|\lambda_s - \mu_s|\bigl|D^\beta g\bigl(n(s), A(s)\bigr)(x)\bigr| + |\mu_s|/m\bigr) \\
&\quad + \sum_{a \in \overline{K}_m} \bigl|\sigma_a(m, \vec{\mu})\bigr|\bigl|D^\beta g\bigl(n, \overline{K}_m \setminus \{a\}\bigr)(x)\bigr|
\end{aligned}$$

$$+\sum\bigl\{|\sigma_u|\bigl|D^\beta g\bigl(n,\overline{K}_m\cup\bigl\{d^{v,m}\colon\ v\in\operatorname{dom}h',\ v\neq u\bigr\}\bigr)(x)\bigr|\colon\ u\in\operatorname{dom}h'\bigr\}\Biggr]$$

$$\leqslant\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}C_H\Biggl[\sum_{s\in\mathcal{G}'}\bigl(|\lambda_s-\mu_s|+|\mu_s|/m\bigr)+\sum_{a\in\overline{K}_m}\bigl|\sigma_a(m,\vec{\mu})\bigr|+\sum_{u\in\operatorname{dom}h'}|\sigma_u|\Biggr],$$

where $C_H=\sup_{x\in L,\,|\alpha|\leqslant i_0}|D^\alpha H(x)|$ and the second and third inequalities used the first part of Proposition 2.3(b), the fact that $n(s)\geqslant 2|A(s)|$, Proposition 2.3(a) and the fact that $n\geqslant 2|\overline{K}_m|$.

We may choose $m\geqslant m_0$, $\mu_s\in\mathbb{Q}$ for each $s\in\mathcal{G}'$ and $r>0$ so that (e), (f) and (g) are satisfied with $b^{x,m}$ in the place of b_x and $d^{u,m}$ in the place of d_u. (First get a neighborhood of $\vec{\lambda}$, an $m_1\geqslant m_0$ and an r so that (e), (g) are satisfied for $\vec{\mu}$ in the given neighborhood of $\vec{\lambda}$ and $m\geqslant m_1$. Then choose such $\vec{\mu}$ and m so that $|\sigma_u(m,\vec{\mu})|<r$ for each $u\in\operatorname{dom}h'$, giving (f).) Note that we have $f'\in N$ because $h, H, K_0, f[K_1], \{b^{x,m}\colon\ x\in K_1\}, \{c_{x,m}\colon\ x\in K_0\}\in N$, each μ_s is rational (and hence belongs to N) and each $\sigma_a(m,\vec{\mu})$, for $a\in\overline{K}_m$, is uniquely determined by the condition that $f'(a)=h(a)$ for $a\in\operatorname{dom}h$, $f'(x)=c_{x,m}$ for $a=x\in K_0$ and $f'(b^{x,m})=f(x)$ for $a=b^{x,m}$ $(x\in K_1)$, and hence belongs to N by elementarity. □

3. The Main Lemma

In this section, we prove the main technical lemma for the oracle-cc iteration. We begin by defining two families of open sets in $\mathbb{R}^{t+1}$.

Let $f:\mathbb{C}^t\to\mathbb{C}$ be such that $f[\mathbb{R}^t]\subseteq\mathbb{R}$, let $\lambda>0$ be a rational number, let $n\in\mathbb{N}$ and let $A\subseteq\mathbb{R}^t$ be a finite set. Let $R\subseteq\mathbb{R}^t$ be an open rectangle of the form $R=\prod_{k<t}(a_k,b_k)$, where a_k and b_k are rational numbers, $a_k<b_k$. Let us call such a rectangle a *rational rectangle*. Define

$$V(f,\lambda,n,A,R)=\bigcup\bigl\{\bigl(f+\mu Hg(n,A)\bigr)\restriction R\colon\ |\mu|<\lambda\bigr\}.$$

Proposition 3.1. *The sets $V(f,\lambda,n,A,R)$ have the following properties*:

(a) *If f is continuous and $g(n,A)$ has no zeros in R, then $V(f,\lambda,n,A,R)$ is open in $\mathbb{R}^{t+1}$.*
(b) *If f is continuous, then*

$$\lim_{\lambda\to 0+}\operatorname{diam}V(f,\lambda,n,A,R)=\operatorname{diam}f\restriction\overline{R}$$

(*where the diameter is calculated with respect to the Euclidean norm on $\mathbb{R}^{t+1}$*).

Proof. (a) Say $g(n,A)$ is positive on R. We have

$$V(f,\lambda,n,A,R)=\bigl\{(u,v)\in\mathbb{R}^t\times\mathbb{R}\colon\ u\in R,\ \bigl(f-\lambda Hg(n,A)\bigr)(u)<v<\bigl(f+\lambda Hg(n,A)\bigr)(u)\bigr\}.$$

We prove (b) by contradiction. If (b) were false, then since $f\restriction R\subseteq V(f,\lambda,n,A,R)$ by definition, it must be that for some $\varepsilon>0$, there is a sequence of numbers $\lambda_i>0$ such that $\lambda_i\to 0$ and for each i

$$\operatorname{diam}V(f,\lambda_i,n,A,R)>\varepsilon+\operatorname{diam}f\restriction\overline{R}.$$

Choose u_i,u_i',μ_i,μ_i' so that $u_i,u_i'\in R$, $|\mu_i|<\lambda_i$, $|\mu_i'|<\lambda_i$ and

$$\bigl|\bigl(u_i,\bigl(f+\mu_iHg(n,A)\bigr)(u_i)\bigr)-\bigl(u_i',\bigl(f+\mu_i'Hg(n,A)\bigr)(u_i')\bigr)\bigr|>\varepsilon+\operatorname{diam}f\restriction\overline{R}.$$

Passing to a subsequence, we get $u_i\to u$, $u_i'\to u'$ for some $u,u'\in\overline{R}$. Taking the limit of the above inequality then gives

$$\bigl|\bigl(u,f(u)\bigr)-\bigl(u',f(u')\bigr)\bigr|\geqslant\varepsilon+\operatorname{diam}f\restriction\overline{R}>\operatorname{diam}f\restriction\overline{R},$$

a contradiction. □

When $f:\mathbb{C}^t\to\mathbb{C}$ is a function such that $f[\mathbb{R}^t]\subseteq\mathbb{R}$, $\lambda>0$ is rational, $n\in\mathbb{N}$, $K\subseteq\mathbb{R}^t$ is a nonempty finite set, $K'\subseteq K$ and R_u for $u\in K'$ are rational rectangles, let us write $W(f,\lambda,n,K,K',(R_u\colon\ u\in K'))$ for the set

$$\Big\{(x_u, y_u)_{u\in K'} \in \prod_{u\in K'} (R_u \times \mathbb{R})\colon \text{ for some scalars } \sigma_v \in \mathbb{R} \text{ such that } |\sigma_v| < \lambda \text{ for } v \in K',$$
$$\text{and for all } u \in K',\ \Big(f + \sum_{v\in K'} \sigma_v Hg\big(n, K \setminus \{v\}\big)\Big)(x_u) = y_u\Big\}.$$

Proposition 3.2. *If f is continuous and for any choice of a vector $\bar{x} = (x_u\colon u \in K') \in \prod_{u\in K'} R_u$, the matrix*

$$M(\bar{x}) = \big(g\big(n, K \setminus \{v\}\big)(x_u)\big)_{u,v\in K'}$$

is invertible, then $W(f, \lambda, n, K, K', (R_u\colon u \in K'))$ is open in $(\mathbb{R}^t \times \mathbb{R})^{K'}$.

Proof. Note that invertibility of $M(\bar{x})$ ensures invertibility of $M'(\bar{x})$ as well, where

$$M'(\bar{x}) = \big(H(x_u)g\big(n, K \setminus \{v\}\big)(x_u)\big)_{u,v\in K'},$$

since each row of $M'(\bar{x})$ is a nonzero multiple of the corresponding row of $M(\bar{x})$. Hence the systems of equations in the variables σ_u, $u \in K'$, of the form

$$\Big(f + \sum_{v\in K'} \sigma_v Hg\big(n_1, K \setminus \{v\}\big)\Big)(x_u) = y_u, \quad u \in K'$$

in the definition of $W(f, \lambda, n, K, K', (R_u\colon u \in K'))$ have unique solutions. Note that $M'(\bar{x})$ varies continuously with $\bar{x}$ and hence so does $M'(\bar{x})^{-1}$. The continuity of the map $(x_u, y_u)_{u\in K'} \mapsto (\sigma_u)_{u\in K'}$ gives the desired conclusion. □

Note that the first of these two families of open sets corresponds to the special case of the second where K' is a singleton.

Before stating the main lemma, we recall the definition of oracle-cc forcing.

Definition 3.3. A sequence $\overline{M} = \langle M_\delta\colon \delta < \omega_1\rangle$ is called an *oracle* if each M_δ is a countable transitive model of a sufficiently large fragment of ZFC, $\delta \in M_\delta$ is countable in M_δ and for each $A \subseteq \omega_1$, $\{\delta\colon A \cap \delta \in M_\delta\}$ is stationary in ω_1.

The existence of an oracle is equivalent to $\diamondsuit$ (see [9, Theorem II 7.14]) and hence implies CH. Associated with an oracle $\overline{M}$, there is a filter $\mathrm{Trap}(\overline{M})$ generated by the sets

$$\{\delta < \omega_1\colon A \cap \delta \in M_\delta\}, \quad A \subseteq \omega_1.$$

This is a proper normal filter containing all closed unbounded sets.

Definition 3.4. A partial order P *satisfies the $\overline{M}$-chain condition*, or simply *is $\overline{M}$-cc*, if there is a one-to-one function $f : P \to \omega_1$ such that the set of limit ordinals $\delta < \omega_1$ such that every predense subset of $f^{-1}(\delta)$ of the form $f^{-1}[A]$, where $A \subseteq \delta$ and $A \in M_\delta$, is predense in P belongs to $\mathrm{Trap}(\overline{M})$.

It is not hard to verify that if P is $\overline{M}$-cc, then P is ccc. Also, any one-to-one function $g : P \to \omega_1$ can replace f in the definition.

Lemma 3.5. *Let $\overline{M} = \langle M_\delta\colon \delta < \omega_1\rangle$ be an oracle. Suppose we are given a collection $\mathcal{C}$ of everywhere second category subsets of $\mathbb{R}^{t+1}$, $|\mathcal{C}| = \omega_1$, a countable dense set $A \subseteq \mathbb{R}^t$, countable dense sets $B_x \subseteq \mathbb{R}$ for $x \in A$, a nondecreasing sequence $\{c_i\}_{i<\omega}$ of nonnegative real numbers with $\lim c_i = \infty$, and a strictly positive continuous function $\eta : \mathbb{R}^t \to \mathbb{R}$. Then there is a forcing notion P satisfying the $\overline{M}$-cc such that for every $G \subseteq P$ generic over V, in $V[G]$ there is an entire function $f : \mathbb{C}^t \to \mathbb{C}$ such that $f[\mathbb{R}^t] \subseteq \mathbb{R}$ and*

(i) *$f(x) \in B_x$ for each $x \in A$,*
(ii) *$f \cap C$ is everywhere second category relative to f for every $C \in \mathcal{C}$,*

(iii) *For each* $i < \omega$, *for each multiindex* ζ *such that* $|\zeta| \leqslant i$ *and for each* $x \in \mathbb{R}^t$ *such that* $|x| \geqslant c_i$, $|D^\zeta f(x)| \leqslant \eta(x)$.

Furthermore, if $A' \subseteq A$ *is a dense ground model set and for some* B *we have* $B_x = B$ *for all* $x \in A'$ *and* $B \cap B_x = \emptyset$ *for all* $x \in A \setminus A'$, *then* $f[A']$ *is an interval of* B.

Proof. By replacing $\eta(x)$ by $\min\{1, \eta(x)\}$, we may assume that for all $x \in \mathbb{R}$, $\eta(x) \leqslant 1$. Let H be as in Proposition 2.1 for $\{c_i\}_{i<\omega}$ and η. For the rest of the proof, fix a suitably large regular cardinal θ. Let $\mathcal{B}$ be a countable base of nonempty open sets for $\mathbb{R}^{t+1}$. For bookkeeping purposes, fix a function

$$\gamma : \omega_1 \to \mathcal{B} \times \mathcal{C}$$

so that $\gamma^{-1}(u)$ is uncountable for every $u \in \mathcal{B} \times \mathcal{C}$. We will inductively define partial orders $P(\alpha)$, $\alpha \leqslant \omega_1$, from the following class of partial orders.

Definition 3.6. Let $N \prec H_\theta$. Let $\bar{a} = \langle a_\xi \colon \xi < \alpha \rangle$ be a one-to-one sequence of elements of $\mathbb{R}^{t+1}$, $\alpha \leqslant \omega_1$. $P(\bar{a}, N)$, denotes the partial order consisting of conditions $p = (h_p, f_p, \varepsilon_p, n_p)$ such that

(i) h_p is a finite partial function from $\mathbb{R}^t$ to $\mathbb{R}$
(ii) for each $x \in \operatorname{dom} h_p$, either $x \in A$ and $h_p(x) \in B_x$, or $(x, h_p(x)) = a_\beta$ for some $\beta < \alpha$
(We identify $\mathbb{R}^t \times \mathbb{R}$ with $\mathbb{R}^{t+1}$.)
(iii) $f_p \in (H \operatorname{span} \mathcal{G}_0) \cap N$
(iv) $h_p \subseteq f_p$
(v) for all $i < \omega$, for all multiindices ζ such that $|\zeta| \leqslant i$, for all $x \in \mathbb{R}^t$ such that $|x| \geqslant c_i$, $|D^\zeta f_p(x)| < (1 - \varepsilon_p)\eta(x)$
(vi) ε_p is a rational number, $0 < \varepsilon_p < 1$, and $n_p < \omega$.

The order is given by $p \leqslant q$ if and only if

(vii) $h_p \supseteq h_q$, $n_p \geqslant n_q$
(viii) for each $z \in \mathbb{C}^t$ such that $|z| \leqslant n_q$, $|f_p(z) - f_q(z)| + \varepsilon_p \leqslant \varepsilon_q$
(This gives in particular $\varepsilon_p \leqslant \varepsilon_q$.)

This order relation is transitive because if $p \leqslant q \leqslant r$, then $h_p \supseteq h_q \supseteq h_r$, $n_p \geqslant n_q \geqslant n_r$ and for $z \in \mathbb{C}^t$ such that $|z| \leqslant n_r$,

$$\left|f_p(z) - f_r(z)\right| + \varepsilon_p \leqslant \left|f_q(z) - f_r(z)\right| + \left(\left|f_p(z) - f_q(z)\right| + \varepsilon_p\right) \leqslant \left|f_q(z) - f_r(z)\right| + \varepsilon_q \leqslant \varepsilon_r.$$

We inductively define a one-to-one sequence $\bar{a} = \langle a_\alpha \colon \alpha < \omega_1 \rangle$ of points in $\mathbb{R}^{t+1} \setminus (A \times \bigcup_{x \in A} B_x)$, a sequence of functions $\langle e_\alpha \colon \alpha < \omega_1 \rangle$, a continuous $\in$-increasing sequence $\langle N_\alpha \colon \alpha < \omega_1 \rangle$ of countable elementary submodels of H_θ and a second sequence $\langle N'_\alpha \colon \alpha < \omega_1 \rangle$ of countable elementary submodels of H_θ. Then, for each $\alpha \leqslant \omega_1$, we define

$$P(\alpha) = P(\bar{a} \restriction \alpha, N_\alpha).$$

(The letter P is doing double duty here, but this should not cause any confusion.)

We will arrange that the following conditions hold for all $\alpha < \omega_1$.

(1) N_α and N'_α are countable elementary submodels of H_θ with $\langle a_\beta \colon \beta < \alpha \rangle, N_\alpha, \bigcup_{\beta \leqslant \alpha} M_\beta, e_\alpha \in N'_\alpha$.
(2) $\langle B_x \colon x \in A \rangle$, $\langle c_i \colon i < \omega \rangle$, η, H and $\mathcal{C}$ are elements of N_0.
(3) If α is a limit ordinal, then $N_\alpha = \bigcup_{\beta < \alpha} N_\beta$.
(4) If α is a successor ordinal, then $\langle a_\beta \colon \beta < \alpha \rangle$ and $\langle N'_\beta \colon \beta < \alpha \rangle$ are elements of N_α.
(5) $a_\alpha \in \mathbb{R}^{t+1}$ is a Cohen real over N'_α which belongs to $B \cap C$, where $\gamma(\alpha) = (B, C)$.
(6) e_α is a bijective map of $P(\alpha)$ onto $\omega(1 + \alpha)$.
(7) For each $\beta < \alpha$, $e_\beta \subseteq e_\alpha$.
(8) The predense subsets of $P(\alpha)$ which have the form $e_\alpha^{-1}[S]$ for some $S \subseteq \omega(1 + \alpha)$ such that $S \in \bigcup_{\beta \leqslant \alpha} M_\beta$ are predense in $P(\alpha + 1)$.

Remark 3.7. (a) It follows inductively from (1), (3) and (4) that $a_\beta \in N_\alpha$ whenever $\beta < \alpha$. Together with (2), this gives $h_p \in N_\alpha$ for each $p \in P(\alpha)$.

(b) By (1), (2), (3) and (4), N'_α contains all the parameters $\langle a_\beta\colon \beta < \alpha\rangle$, N_α, H, η, $\langle c_i\colon i < \omega\rangle$ and $\langle B_x\colon x \in A\rangle$ in the definition of $P(\alpha)$.

(c) From (1), (4) and (5) it follows that for any sequence $\alpha_1 < \cdots < \alpha_n < \omega_1$ we have $N'_{\alpha_1} \subseteq \cdots \subseteq N'_{\alpha_n}$, and for each $i = 1, \ldots, n$, a_{α_i} is Cohen generic over N'_{α_i} and for each $i = 1, \ldots, n-1$, $a_{\alpha_i} \in N'_{\alpha_{i+1}}$. Hence $(a_{\alpha_1}, \ldots, a_{\alpha_n})$ is a Cohen generic element of $(\mathbb{R}^{t+1})^n$ over N'_{α_1}.

(d) Set $P = P(\langle a_\alpha\colon \alpha < \omega_1\rangle, \bigcup_{\alpha<\omega_1} N_\alpha)$. Note that the second coordinate contains all entire functions since the M_β's form a $\diamondsuit$ sequence and for each β, $M_\beta \subseteq N'_\beta \subseteq N_{\beta+1}$. The conditions (6)–(8) ensure that P is $\overline{M}$-cc. To see this, let $e = \bigcup_{\alpha<\omega_1} e_\alpha : P \to \omega_1$. For any $\alpha < \omega_1$ we have $e^{-1}[\omega(1+\alpha)] = P(\alpha)$ and for each $S \subseteq \omega(1+\alpha)$ belonging to M_α, whenever $E = e^{-1}[S] = e_\alpha^{-1}[S]$ is predense in $P(\alpha)$, a simple induction on α' using (8) shows that if $\alpha \leqslant \alpha' < \omega_1$, then E is predense in $P(\alpha')$. Thus, E is predense in P. For a club of $\alpha < \omega_1$ we have $\omega(1+\alpha) = \alpha$, so this shows that P satisfies the $\overline{M}$-cc.

To do this, we first arrange (1)–(7) inductively. At stage α, we define first N_α then e_α then N'_α then a_α. N_α is given by (1) and (2) if $\alpha = 0$, and by (1) and either (3) or (4) if $\alpha > 0$. The definition of e_α when α is a limit ordinal is dictated by (7) since $P(\alpha) = \bigcup_{\beta<\alpha} P(\beta)$ in this case. We take e_0 to be any function satisfying (6), and if $\alpha = \beta + 1$, then e_α is any extension of e_β satisfying (6). Finally, N'_α is given by (1) and a_α is given by (5).

We must check that the construction gives (8). Let $E = e_\alpha^{-1}[S]$ be a predense subset of $P(\alpha)$, where $S \subseteq \omega(1+\alpha)$, $S \in \bigcup_{\beta\leqslant\alpha} M_\beta$. Note that (1) implies $E \in N'_\alpha$. We will show that E remains predense in $P(\alpha+1)$. Let

$$p \in P(\alpha+1) \setminus P(\alpha).$$

Let $h = h_p \setminus \{a_\alpha\}$. Then

$$h \subseteq \left(\bigcup_{x\in A} \{x\} \times B_x\right) \cup \{a_\beta\colon \beta < \alpha\}$$

and $f_p \in (H \operatorname{span} \mathcal{G}_0) \cap N_{\alpha+1}$. We must show that p is compatible with an element of E.

Write $a_\alpha = (x_\alpha, y_\alpha)$, where $x_\alpha \in \mathbb{R}^t$, $y_\alpha \in \mathbb{R}$.

Case 1. $a_\alpha \notin h_p$.

We have $h_p \in N_\alpha$, so Proposition 2.4 gives a function $f' \in (H \operatorname{span} \mathcal{G}_0) \cap N_\alpha$ such that

1(a) $h_p \subseteq f'$,
1(e) for all $i < \omega$, for all ζ such that $|\zeta| \leqslant i$ and all $x \in \mathbb{R}^t$ such that $|x| \geqslant c_i$, $|D^\zeta f_p(x) - D^\zeta f'(x)| < \frac{1}{2}\varepsilon_p \eta(x)$ (and hence $|D^\zeta f'(x)| < (1 - \frac{1}{2}\varepsilon_p)\eta(x)$),
1(g) for all $z \in \mathbb{C}$ such that $|z| \leqslant n_p$, $|f_p(z) - f'(z)| < \frac{1}{2}\varepsilon_p$.

The letters in the labels here correspond to those in the statement of Proposition 2.4. The number 1 is a reference to case 1. We will use similar notation in the rest of the proof when applying this proposition. (To get f', in Proposition 2.4, take $\varepsilon(x) = \eta(x)$ but get the conclusion for $\frac{1}{2}\varepsilon_p\eta$ instead of η. See Remark 2.5. Note that $\frac{1}{2}\varepsilon_p\eta \leqslant \frac{1}{2}\varepsilon_p$ because $\eta \leqslant 1$.) Then

$$q = \left(h_p, f', \frac{1}{2}\varepsilon_p, n_p\right)$$

belongs to $P(\alpha)$. Also, q and some $r \in E$ have a common extension $q' \in P(\alpha)$. Then $q' \leqslant p$ since for each $z \in \mathbb{C}^t$ such that $|z| \leqslant n_p$, we have also $|z| \leqslant n_q$ (since $n_q = n_p$) and hence

$$\begin{aligned}\left|f_{q'}(z) - f_p(z)\right| + \varepsilon_{q'} &\leqslant \left|f_q(z) - f_p(z)\right| + \left|f_{q'}(z) - f_q(z)\right| + \varepsilon_{q'}\\ &= \left|f'(z) - f_p(z)\right| + \left|f_{q'}(z) - f_q(z)\right| + \varepsilon_{q'}\end{aligned}$$

$$\leqslant \frac{1}{2}\varepsilon_p + \left|f_{q'}(z) - f_q(z)\right| + \varepsilon_{q'}$$
$$\leqslant \frac{1}{2}\varepsilon_p + \varepsilon_q = \varepsilon_p.$$

Case 2. $a_\alpha \in h_p$.

We have $h_p = h \cup \{a_\alpha\}$ and $h \in N_\alpha$. Proposition 2.4 with $h' = \{a_\alpha\}$ gives $n_1 \in \mathbb{N}$, a function $f' \in (H \operatorname{span} \mathcal{G}_0) \cap N_\alpha$ and a rational number $\lambda_0 > 0$ such that

2(a) $h \subseteq f'$,
2(e) $n_1 \geqslant 8|\operatorname{dom} h|$ and for all $i < \omega$, for all ζ such that $|\zeta| \leqslant i$ and all $x \in \mathbb{R}^t$ such that $|x| \geqslant c_i$ and for all $\lambda \in \mathbb{R}$ such that $|\lambda| \leqslant \lambda_0$,

$$\left|D^\zeta f_p(x) - D^\zeta\bigl(f' + \lambda H g(n_1, \operatorname{dom} h)\bigr)(x)\right| < \frac{1}{2}\varepsilon_p \eta(x)$$

(and hence in particular $|D^\zeta(f' + \lambda H g(n_1, \operatorname{dom} h))(x)| < (1 - \frac{1}{2}\varepsilon_p)\eta(x)$),
2(f) $g(n_1, \operatorname{dom} h)(x_\alpha) \neq 0$ and for some number λ such that $|\lambda| < \lambda_0$, we have $h_p \subseteq f' + \lambda H g(n_1, \operatorname{dom} h)$,
2(g) for all $z \in \mathbb{C}^t$ such that $|z| \leqslant n_p$ we have $\lambda_0 |H(z) g(n_1, \operatorname{dom} h)(z)| < \frac{1}{4}\varepsilon_p$ and for all λ such that $|\lambda| < \lambda_0$, $|f_p(z) - (f' + \lambda H g(n_1, \operatorname{dom} h))(z)| < \frac{1}{4}\varepsilon_p$.

Choose a rational rectangle R such that $x_\alpha \in R$ and $g(n_1, \operatorname{dom} h)$ has no zeros in R. Consider the open set $U = V(f', \lambda_0, n_1, \operatorname{dom} h, R)$ in $\mathbb{R}^{t+1}$. (See Proposition 3.1.) By 2(f), $a_\alpha \in U$. Define

$$q_0 = \left(h, f', \frac{1}{2}\varepsilon_p, n_p\right)$$

and notice that $q_0 \in P(\alpha)$.

Claim 3.8. *The union of the open sets $V(f_q, \mu, m, \operatorname{dom} h_q, R')$ such that*

1. *$q \in P(\alpha)$ is a common extension of q_0 and an element of E;*
2. *R' is a rational rectangle, $R' \subseteq R$;*
3. *$g(m, \operatorname{dom} h_q)$ has no zeros in R';*
4. *$\mu > 0$ is rational, $m \in \mathbb{N}$, $m \geqslant 8|\operatorname{dom} h_q|$;*
5. *$V(f_q, \mu, m, \operatorname{dom} h_q, R') \subseteq U$;*
6. *for all $z \in \mathbb{C}$ such that $|z| \leqslant n_q$, $\mu|H(z) g(m, \operatorname{dom} h_q)(z)| \leqslant \frac{1}{2}\varepsilon_q$ and for all $i < \omega$, for all ζ such that $|\zeta| \leqslant i$ and for all $x \in \mathbb{R}^t$ such that $|x| \geqslant c_i$, $\mu|D^\zeta(H g(m, \operatorname{dom} h_q))(x)| < \frac{1}{2}\varepsilon_q \eta(x)$*
   (and hence for all λ such that $|\lambda| \leqslant \mu$ we have $|D^\zeta(f_q + \lambda H g(m, \operatorname{dom} h_q))(x)| < (1 - \frac{1}{2}\varepsilon_q)\eta(x)$) is dense in U.

Proof. Let $(u, v) \in U \cap \bigcup_{x \in A} \{x\} \times B_x$. Let $\delta > 0$ be arbitrary but small enough so that the Euclidean ball $B(u, v, \delta)$ of radius δ centered at (u, v) is contained inside U. Note that the balls of this form are a π-base for U. Hence we shall be done if we show that the $V(f_q, \mu, m, \operatorname{dom} h_q, R')$ in part 5 of the claim can be chosen inside $B(u, v, \delta)$.

Fix λ_1 such that $|\lambda_1| < \lambda_0$ and $\bar{f}'(u) = v$, where $\bar{f}' = f' + \lambda_1 H g(n_1, \operatorname{dom} h)$. By uniqueness of λ_1, λ_1 belongs to N_α. We have that $q_1 \in P(\alpha)$ where

$$q_1 = \left(h \cup \bigl\{(u, v)\bigr\}, \bar{f}', \frac{1}{4}\varepsilon_p, n_p\right).$$

Also, q_1 extends q_0. (Use 2(g).) Choose a common extension $q \in P(\alpha)$ of q_1 and some $r \in E$.

Since $(u, v) \in h_q \subseteq f_q$, we can pick $w \notin \operatorname{dom} h_q$ such that $w \in R$ and $(w, f_q(w)) \in B(u, v, \frac{1}{2}\delta)$. Pick $m \geqslant 8|\operatorname{dom} h_q|$ large enough so that

$$g(m, \operatorname{dom} h_q)(w) \neq 0.$$

Letting $\overline{R}'$ denote the closure of R', pick a rational rectangle R' containing w so that $g(m, \operatorname{dom} h_q)$ has no zeros in R', $f_q \restriction \overline{R}'$ has diameter less than $\frac{1}{2}\delta$ and is contained in $B(u, v, \frac{1}{2}\delta)$. For $\mu > 0$ small enough we have that part 6 of the claim holds.

[For the first assertion of part 6 this is clear. For the second, proceed as follows. By Proposition 2.3(c), whenever $|\zeta'| \leqslant i$ and $|x| \geqslant c_i$ we have $|D^{\zeta'}(Hg(m, \operatorname{dom} h_q))(x)| \leqslant (3/4)^{|\zeta'|}\zeta(x)\eta(x)$. For large enough i_0, we have $(3/4)^{i_0} \leqslant \frac{1}{2}\varepsilon_q$ and hence when $i_0 < |\zeta'| \leqslant i$ and $|x| \geqslant c_i$, $|D^{\zeta'}(Hg(m, \operatorname{dom} h_q))(x)| \leqslant (3/4)^{i_0}\zeta(x)\eta(x) \leqslant \frac{1}{2}\varepsilon_q\eta(x)$. Moreover, for all x outside some closed ball L we have $\zeta(x) < \frac{1}{2}\varepsilon_q$ and hence, when $|\zeta'| \leqslant i$ and $|x| \geqslant c_i$, $|D^{\zeta'}(Hg(m, \operatorname{dom} h_q))(x)| \leqslant (3/4)^{|\zeta'|}\zeta(x)\eta(x) < \frac{1}{2}\varepsilon_q\eta(x)$. For μ small enough we will also have $\mu|D^{\zeta'}(Hg(m, \operatorname{dom} h_q))(x)| < \frac{1}{2}\varepsilon_q\eta(x)$ whenever $|\zeta'| \leqslant i_0$ and $x \in L$, giving part 6 of the claim.]

If μ is small enough, we also have that $V(f_q, \mu, m, \operatorname{dom} h_q, R')$ is contained in $B(u, v, \delta) \subseteq U$. (Use Proposition 3.1(b).)

This proves Claim 3.8. □

The dense open subset of U given by Claim 3.8 belongs to N'_α. By (5), there are q, μ, m, R' satisfying Claim 3.8(1–6) for which $a_\alpha \in V(f_q, \mu, m, \operatorname{dom} h_q, R')$. Choosing λ with $|\lambda| < \mu$ so that $a_\alpha \in f_q + \lambda Hg(m, \operatorname{dom} h_q)$, we get that

$$q' = \left(h_q \cup \{a_\alpha\}, f_q + \lambda Hg(m, \operatorname{dom} h_q), \frac{1}{2}\varepsilon_q, n_q\right)$$

belongs to $P(\alpha + 1)$ (by the second part of clause 6 of Claim 3.8) and extends both q and p. It extends q by the first part of clause 6 of Claim 3.8. To see that $q' \leqslant p$, note that for each $z \in \mathbb{C}$ such that $|z| \leqslant n_p$,

$$\begin{aligned}
\left|f_{q'}(z) - f_p(z)\right| + \varepsilon_{q'} &\leqslant \left|f_q(z) - f_p(z)\right| + \left|f_{q'}(z) - f_q(z)\right| + \varepsilon_{q'} \\
&\leqslant \left|f_q(z) - f_p(z)\right| + \varepsilon_q \\
&\leqslant \left|f_{q_0}(z) - f_p(z)\right| + \left|f_q(z) - f_{q_0}(z)\right| + \varepsilon_q \\
&\leqslant \left|f_{q_0}(z) - f_p(z)\right| + \varepsilon_{q_0} \\
&\leqslant \frac{1}{4}\varepsilon_p + \frac{1}{2}\varepsilon_p < \varepsilon_p.
\end{aligned}$$

Thus, p is compatible with q and hence with some element of E.

This completes the proof of (8).

Now take $P = \bigcup_{\alpha<\omega_1} P(\alpha)$. In $V[G]$, let h be the union of the h_p parts of the conditions in G. Then h is a partial function from $\mathbb{R}^t$ into $\mathbb{R}$. For any $x \in A$, every condition can be extended to one with $x \in \operatorname{dom} h_p$.

[Given $p = (h_p, f_p, \varepsilon_p, n_p)$ such that $x \notin \operatorname{dom} h_p$, get f' from Proposition 2.4 such that $h_p \subseteq f'$, $f'(x) \in B_x$, when $|\zeta| \leqslant i$ and $|x| \geqslant c_i$ we have $|D^\zeta f_p(x) - D^\zeta f'(x)| < \frac{1}{2}\varepsilon_p\eta(x)$, and for $|z| \leqslant n_p$ we have $|f_p(z) - f'(z)| < \frac{1}{2}\varepsilon_p$. Then $q = (h_p \cup \{(x, f'(x))\}, f', \frac{1}{2}\varepsilon_p, n_p)$ is the desired extension of p.]

Hence $A \subseteq \operatorname{dom} h$. Suppose $A' \in V$ is a dense subset of A such that for some set B we have that for all $x \in A'$, $B_x = B$, and for all $x \in A \setminus A'$, $B_x \cap B = \emptyset$. We want to show that in $V[G]$, $h[A']$ is an interval of B. Let $x_1, x_2 \in A'$ and let $y \in B$ satisfy $h(x_1) < y < h(x_2)$. By what we just proved, the conditions p with $x_1, x_2 \in \operatorname{dom} h_p$ are dense. Fix such a condition $p_0 \in G$ and consider any $p \leqslant p_0$. We have $f_p(x_1) = h_p(x_1) = h(x_1) < y < h(x_2) = h_p(x_2) = f_p(x_2)$. On the straight line segment joining x_1 to x_2, there is a point x such that $f_p(x) = y$. If $x \in \operatorname{dom} h_p$, then $(x, y) \in h_p$ and because no a_α has an element of B as a second coordinate, necessarily $x \in A$ and $y \in B_x$. By the assumption about B, since $y \in B$ we have $x \in A'$. Hence p forces $y \in h[A']$. Now suppose that $x \notin \operatorname{dom} h_p$. Taking $K_1 = \{x\}$ in Proposition 2.4, get f' and $a \in A'$ such that $h_p \subseteq f'$, $f'(a) = f_p(x) = y$, when $|\zeta| \leqslant i$ and $|x| \geqslant c_i$ we have $|D^\zeta f_p(x) - D^\zeta f'(x)| < \frac{1}{2}\varepsilon_p\eta(x)$, and for $|z| \leqslant n_p$ we have $|f_p(x) - f'(z)| < \frac{1}{2}\varepsilon_p$. Then $q = (h_p \cup \{(a, y)\}, f', \frac{1}{2}\varepsilon_p, n_p)$ forces $y \in h[A']$.

We get f as follows. For $k \in \mathbb{N}$, choose $p_k \in G$ such that $p_{k+1} \leqslant p_k$, $n_{p_k} \geqslant k$ and $\varepsilon_{p_k} < 1/k$. The sequence $\{f_{p_k}\}$ is uniformly Cauchy on compact sets because for $\ell > k$ and $|z| \leqslant k (\leqslant n_{p_k})$, $|f_{p_\ell}(z) - f_{p_k}(z)| \leqslant \varepsilon_{p_k} < 1/k$. (By f_p here we mean the entire function with the same code as the ground model f_p. Note that entire functions have canonical codes, namely the (multiindexed) sequence of coefficients for the power series.) Define $f : \mathbb{C}^t \to \mathbb{C}$ by

$f(z) = \lim_{k\to\infty} f_{p_k}(z)$. For all ζ, we have $D^\zeta f(z) = \lim_{k\to\infty} D^\zeta f_{p_k}(z)$ uniformly on compact sets [7, Theorem I.4, p. 6]. Also, for each $a \in \operatorname{dom} h$, we can choose k such that $|a| \leqslant n_{p_k}$, and $p \in G$ such that $p \leqslant p_k$ and $a \in \operatorname{dom} h_p$. Then

$$\left|f_{p_k}(a) - h(a)\right| = \left|f_{p_k}(a) - h_p(a)\right| = \left|f_{p_k}(a) - f_p(a)\right| \leqslant \varepsilon_{p_k} < 1/k$$

and hence $f(a) = \lim_{k\to\infty} f_{p_k}(a) = h(a)$. Thus, $h \subseteq f$. Also, for each $i < \omega$, for each ζ such that $|\zeta| \leqslant i$, and for each $x \in \mathbb{R}^t$ such that $|x| \geqslant c_i$,

$$\left|D^\zeta f(x)\right| = \lim_{k\to\infty} \left|D^\zeta f_{p_k}(x)\right| \leqslant \eta(x).$$

By Remark 3.7(d), P is $\overline{M}$-cc.

Now fix $C \in \mathcal{C}$. We must check that in V^P we have that $Y = \{x \in \operatorname{dom} h\colon (x, h(x)) \in C\}$ is everywhere second category in $\mathbb{R}^t$. Suppose it is first category in some rational rectangle R_0. Let U_i, $i < \omega$, be dense open subsets of R_0 such that $Y \cap \bigcap_{i<\omega} U_i = \emptyset$. Then for some $p \in G_P$ and names $\dot{U}_i$ for the sets U_i, we have

$$p \Vdash_P Y \cap \bigcap_{i<\omega} \dot{U}_i = \emptyset.$$

Let $\mathcal{R}$ be the collection of all rational rectangles and, for each $i < \omega$, let $\langle S^i_R\colon R \in \mathcal{R}\rangle$ be such that for each $R \in \mathcal{R}$, S^i_R is an antichain of P and

$$\Vdash_P \ \forall R \in \mathcal{R} \ \left(R \subseteq \dot{U}_i \iff \dot{G}_P \cap S^i_R \neq \emptyset\right).$$

Pick $w_0 \in R_0 \setminus \operatorname{dom} h_p$. Let $m \geqslant 8|\operatorname{dom} h_p|$ be large enough so that $g(m, \operatorname{dom} h_p)(w_0) \neq 0$. Fix a rational rectangle R_1 such that $w_0 \in R_1 \subseteq R_0$, $R_1 \cap \operatorname{dom} h_p = \emptyset$ and $g(m, \operatorname{dom} h_p)$ has no zeros in R_1. Choose a rational number $\lambda_0 > 0$ such that

(i) for all $z \in \mathbb{C}^t$ such that $|z| \leqslant n_p$, $\lambda_0 |H(z) g(m, \operatorname{dom} h_p)(z)| < \frac{1}{2}\varepsilon_p$
(ii) for all $i < \omega$, for all ζ such that $|\zeta| \leqslant i$ and for all $x \in \mathbb{R}^t$ such that $|x| \geqslant c_i$, $\lambda_0 |D^\zeta (Hg(m, \operatorname{dom} h_p))(x)| < \frac{1}{2}\varepsilon_p \eta(x)$

and hence for each $\lambda \in \mathbb{R} \cap \bigcup_{\alpha<\omega_1} N_\alpha$ (which equals $\mathbb{R}$ since $M_\alpha \subseteq N_{\alpha+1}$ for each $\alpha < \omega_1$ and $\langle M_\alpha : \alpha < \omega_1\rangle$ is a $\diamondsuit$ sequence) such that $|\lambda| < \lambda_0$,

$$\left(h_p, f_p + \lambda H g(m, \operatorname{dom} h_p), \frac{1}{2}\varepsilon_p, n_p\right)$$

belongs to P and is an extension of p. (For (ii) see the proof of part 6 of Claim 3.8.)

Let $B \in \mathcal{B}$ be contained in the open set $V(f_p, \lambda_0, m, \operatorname{dom} h_p, R_1)$. Choose any $\alpha < \omega_1$ for which

$$\left\langle S^i_R\colon i < \omega,\ R \in \mathcal{R}\right\rangle, \quad \langle \dot{U}_i\colon i < \omega\rangle \in N_\alpha \text{ and } p \in P(\alpha) \subseteq N_\alpha \text{ and } \gamma(\alpha) = (B, C).$$

[For some $\alpha_0 < \omega_1$, we have $\langle S^i_R\colon i < \omega,\ R \in \mathcal{R}\rangle, \langle \dot{U}_i\colon i < \omega\rangle \in M_{\alpha_0}$. Then any $\alpha > \alpha_0$ satisfies the first two requirements.]

By (5), $a_\alpha = (x_\alpha, y_\alpha)$ is a Cohen real over $N'_\alpha \supset N_\alpha$ and belongs to $B \cap C$.

Note: $(x_\alpha, y_\alpha) \in B \cap C \subseteq B \subseteq V(f_p, \lambda_0, m, \operatorname{dom} h_p, R_1)$ and the latter set is the union of restrictions to R_1 of various functions, so $x_\alpha \in R_1$.

For a suitable value of λ_1 such that $|\lambda_1| < \lambda_0$, we have that $\{a_\alpha\} \subseteq f_p + \lambda_1 g(m, \operatorname{dom} h_p)$ and

$$p_1 = \left(h_p \cup \{a_\alpha\}, f_p + \lambda_1 g(m, \operatorname{dom} h_p), \frac{1}{2}\varepsilon_p, n_p\right)$$

extends p in P. (Note that $\lambda_1 \in N_{\alpha+1}$.) Since $a_\alpha \in C$, we have $p_1 \Vdash x_\alpha \in Y \cap R_0$ and hence

$$p_1 \Vdash x_\alpha \notin \bigcap_{i<\omega} \dot{U}_i.$$

For some $p_2 \leqslant p_1$ and some $i < \omega$, we have

$$p_2 \Vdash x_\alpha \notin \dot{U}_i.$$

Let $\tilde{h}_0 = h_{p_2} \upharpoonright N_\alpha$ and let $\tilde{h}_1 = h_{p_2} \setminus \tilde{h}_0$.

By Proposition 2.4, there are a function $f' \in (H \operatorname{span} \mathcal{G}_0) \cap N_\alpha$, distinct points $d_u \notin \operatorname{dom} \tilde{h}_0$, $u \in \operatorname{dom} \tilde{h}_1$, a positive integer n_1, and a positive rational number μ_0 so that we have, letting $K = \operatorname{dom} \tilde{h}_0 \cup \{d_u\colon u \in \operatorname{dom} \tilde{h}_1\}$,

∗(a) $\tilde{h}_0 \subseteq f'$,
∗(d) for each $u \in \operatorname{dom} \tilde{h}_1$, $d_u \in \mathbb{Q}^t$,
∗(e) $n_1 \geqslant 8|K|$ and for all $i < \omega$, for all ζ such that $|\zeta| \leqslant i$ and for all $\sigma_v \in \mathbb{R}$ such that $|\sigma_v| \leqslant \mu_0$, $v \in \operatorname{dom} \tilde{h}_1$, we have that for $x \in \mathbb{R}^t$ such that $|x| \geqslant c_i$,

$$\left| D^\zeta f_{p_2}(x) - D^\zeta \left(f' + \sum_{v \in \operatorname{dom} \tilde{h}_1} \sigma_v H g\big(n_1, K \setminus \{d_v\}\big) \right)(x) \right| < \frac{1}{2} \varepsilon_{p_2} \eta(x),$$

∗(f) the matrix $M = (g(n_1, K \setminus \{d_v\})(u))_{u,v \in \operatorname{dom} \tilde{h}_1}$ is invertible, and for some real numbers σ_v such that $|\sigma_v| < \mu_0$, $v \in \operatorname{dom} \tilde{h}_1$, we have

$$h_{p_2} \subseteq f' + \sum_{v \in \operatorname{dom} \tilde{h}_1} \sigma_v H g\big(n_1, K \setminus \{d_v\}\big),$$

∗(g) for all $z \in \mathbb{C}^t$ such that $|z| \leqslant n_{p_2}$ and for all $\sigma_v \in \mathbb{R}$ such that $|\sigma_v| \leqslant \mu_0$, $v \in \operatorname{dom} \tilde{h}_1$,

$$\mu_0 \sum_{v \in \operatorname{dom} \tilde{h}_1} \big| H(z) g\big(n_1, K \setminus \{d_v\}\big)(z) \big| < \frac{1}{4} \varepsilon_{p_2}$$

and $|f_{p_2}(z) - f'(z)| < \frac{1}{2} \varepsilon_{p_2}$.

For $u \in \operatorname{dom} \tilde{h}_1$, choose pairwise disjoint rational rectangles R_u disjoint from $\operatorname{dom} \tilde{h}_0$ and such that $R_{x_\alpha} \subseteq R_1$, for all $u \in \operatorname{dom} \tilde{h}_1$, $u \in R_u$, and for any choice of a vector $\bar{b} = (b_u\colon u \in \operatorname{dom} \tilde{h}_1) \in \prod_{u \in \operatorname{dom} \tilde{h}_1} R_u$, the matrix

$$M(\bar{b}) = \big(g\big(n_1, K \setminus \{d_v\}\big)(b_u)\big)_{u,v \in \operatorname{dom} \tilde{h}_1}$$

is invertible. Let U be the open subset of $(\mathbb{R}^{t+1})^{K'}$ given by

$$U = W\big(f', \mu_0, n_1, K, K', (R_u\colon u \in K')\big),$$

where $K' = \{d_u\colon u \in \operatorname{dom} \tilde{h}_1\}$ and R_{d_u} denotes the same rectangle as R_u for each $u \in \operatorname{dom} \tilde{h}_1$. (The switch from $\operatorname{dom} \tilde{h}_1$ to K' as an index set is important because we want $U \in N_\alpha$.) U is open by Proposition 3.2.

By ∗(f), $((x, h_{p_2}(x))\colon x \in \operatorname{dom} \tilde{h}_1)$ belongs to U modulo identifying $\operatorname{dom} \tilde{h}_1$ with K' via $u \mapsto d_u$. Let q_0 be the member of $P(\alpha)$ defined by

$$q_0 = \left(\tilde{h}_0, f', \frac{1}{2} \varepsilon_{p_2}, n_{p_2} \right).$$

(Take $\sigma_v = 0$ in ∗(e) to see that Definition 3.6(v) holds.)

Claim 3.9. *The union of the open sets $W(f_q, \mu, n_1, K, K', (R'_u\colon u \in K'))$ such that*

1. *for each $u \in K'$, R'_u is a rational rectangle, $R'_u \subseteq R_u$,*
2. *$q \in P(\alpha)$ is a common extension of q_0 and an element of S^i_R for some $R \in \mathcal{R}$ such that $R'_{d_{x_\alpha}} \subseteq R$,*
3. *$\mu > 0$ is rational,*
4. *$W(f_q, \mu, n_1, K, K', (R'_u\colon u \in K')) \subseteq U$,*
5. *for all $z \in \mathbb{C}$ such that $|z| \leqslant n_q$, $\mu \sum\{|H(z) g(n_1, K \setminus \{v\})(z)|\colon v \in K'\} \leqslant \frac{1}{2}\varepsilon_q$ and for all $i < \omega$, for all ζ such that $|\zeta| \leqslant i$ and for all $x \in \mathbb{R}^t$ such that $|x| \geqslant c_i$,*

$$\mu \sum \big\{ \big| D^\zeta \big(H g\big(n_1, K \setminus \{v\}\big)\big)(x) \big| \colon v \in K' \big\} < \frac{1}{2} \varepsilon_q \eta(x)$$

(*and hence for all* λ_v *such that* $|\lambda_v| \leqslant \mu$ *for* $v \in K'$, *we have* $|D^\zeta(f_q + \sum_{v\in K'} \lambda_v Hg(n_1, K \setminus \{v\})(x)| < (1 - \frac{1}{2}\varepsilon_q)\eta(x)$)*,*

is dense in U.

Proof. Fix any

$$((b_u, c_u))_{u\in K'} \in U \cap \left(\mathbb{Q}^t \times \mathbb{Q}\right)^{K'}$$

and fix any $\delta > 0$ small enough so that the ball B_δ of radius δ centered at $((b_u, c_u))_{u\in K'}$ is contained inside U. It will suffice to find q satisfying 1–5 and such that the set $W(f_q, \mu, n_1, K, K', (R'_u\colon u \in K'))$ from 4 is contained in B_δ.

Fix λ'_v, for $v \in K'$, such that $|\lambda'_v| < \mu_0$ and $f_{q_1}(b_u) = c_u$ for each $u \in K'$, where

$$f_{q_1} = f' + \sum \left\{\lambda'_v Hg\left(n_1, K \setminus \{v\}\right)\colon v \in K'\right\}.$$

(We will define a condition q_1 shortly.) By invertibility of $M'(\bar b)$, $\bar b = (b_u\colon u \in K')$, the λ'_v are uniquely determined and hence are in N_α. Thus, $f_{q_1} \in N_\alpha$.

Let $M \in \mathbb{N}$ satisfy $M \geqslant \sup\{|x|\colon x \in \prod_{u\in K'} R_u\}$.

Choose rational numbers δ_0, μ'_0 such that $\delta_0 > 0$, $0 < \mu'_0 < \mu_0$ and, for each $u \in K'$, choose $e_u \in \mathbb{Q}^t$ such that for each $k < t$, $(b_u)_k < (e_u)_k$, and letting $[b_u, e_u]$ denote the set $\prod_{k<t}[(b_u)_k, (e_u)_k]$, we have $[b_u, e_u] \subseteq R_u$ and for any function $f \in H \operatorname{span} \mathcal{G}_0$ such that for each $z \in \mathbb{C}^t$ such that $|z| \leqslant M$, $|f(z) - f_{q_1}(z)| < \delta_0$ we have, letting (b_u, e_u) denote the rational rectangle $\prod_{k<t}((b_u)_k, (e_u)_k)$,

$$W\left(f, \mu'_0, n_1, K, K', \left((b_u, e_u)\colon u \in K'\right)\right) \subseteq B_\delta. \qquad (*)$$

[For all large enough $m \in \mathbb{N}$, setting $\delta_0 = 1/m$, $\mu'_0 = 1/m$ and $e_u = e^m_u$, where $(e^m_u)_k = (b_u)_k + (1/m)$ for $k < t$, works. Otherwise, for infinitely many—for notational convenience, say all—$m \in \mathbb{N}$, $(*)$ fails for some function $f = f_m \in H \operatorname{span} \mathcal{G}_0$ such that $|f_m(z) - f_{q_1}(z)| \leqslant 1/m$ for all $z \in \mathbb{C}^t$ such that $|z| \leqslant M$, when $\mu'_0 = 1/m$ and $e_u = e^m_u$. Let W_m denote the set on the left-hand side of $(*)$ with these values of the parameters. Pick $((x^m_u, y^m_u))_{u\in K'} \in W_m \setminus B_\delta$. Let σ^m_v, $v \in K'$, be scalars such that $|\sigma^m_v| < 1/m$ and

$$\left(f_m + \sum_{v\in K'} \sigma^m_v Hg\left(n_1, K \setminus \{v\}\right)\right)\left(x^m_u\right) = y^m_u$$

for each $u \in K'$. For each $u \in K'$, we have $x^m_u \in (b_u, e^m_u)$ and hence $x^m_u \to b_u$ as $m \to \infty$. Since the functions $Hg(n_1, K \setminus \{v\})$ are bounded on $\mathbb{R}^t$, we have that for all $u, v \in K'$, $\sigma^m_v H(x^m_u) g(n_1, K \setminus \{v\})(x^m_u) \to 0$ as $m \to \infty$. Moreover, as $m \to \infty$,

$$f_m\left(x^m_u\right) = f_{q_1}(b_u) + \left[f_{q_1}\left(x^m_u\right) - f_{q_1}(b_u)\right] + \left[f_m\left(x^m_u\right) - f_{q_1}\left(x^m_u\right)\right] \to f_{q_1}(b_u) = c_u.$$

Thus,

$$y^m_u = \left(f_m + \sum_{v\in K'} \sigma^m_v Hg\left(n_1, K \setminus \{v\}\right)\right)\left(x^m_u\right) \to c_u$$

as $m \to \infty$. This gives

$$\left(\left(x^m_u, y^m_u\right)\right)_{u\in K'} \to ((b_u, c_u))_{u\in K'} \in B_\delta,$$

which contradicts the fact that $((x^m_u, y^m_u))_{u\in K'} \notin B_\delta$ for each $m \in \mathbb{N}$.]

Let

$$q_1 = \left(\tilde h_0, f_{q_1}, \min\left(\frac{1}{2}\delta_0, \frac{1}{4}\varepsilon_{p_2}\right), \max(n_{p_2}, M)\right).$$

We have $q_1 \in P(\alpha)$ and $q_1 \leqslant q_0$. (Use $*$(e) to check Definition 3.6(v) and use $*$(g) to check Definition 3.6(viii).) Choose an extension $q' \in P$ of q_1 which also extends a member of S^i_R for some $R \in \mathcal{R}$ with $R \subseteq (b_{d_{x_\alpha}}, e_{d_{x_\alpha}}) \subseteq R_{x_\alpha}$. (Extend q' to decide a rational rectangle R contained in both $(b_{d_{x_\alpha}}, e_{d_{x_\alpha}}) \subseteq R_{x_\alpha} \subseteq R_1 \subseteq R_0$ and the dense open set $\dot U_i \subseteq R_0$. Then extend q' further so it extends a member of S^i_R.) We need to approximate $f_{q'}$ in N_α. Get $f'' \in (H \operatorname{span} \mathcal{G}_0) \cap N_\alpha$ from Proposition 2.4 such that

3.9(a) $h_{q'} \restriction N_\alpha \subseteq f''$,
3.9(e) for all $i < \omega$, for all ζ such that $|\zeta| \leqslant i$ and for all $x \in \mathbb{R}^t$ such that $|x| \geqslant c_i$,

$$\left|D^\zeta f_{q'}(x) - D^\zeta f''(x)\right| < \frac{1}{2}\varepsilon_{q'}\eta(x),$$

3.9(g) for all $z \in \mathbb{C}^t$ such that $|z| \leqslant n_{q'}$, $|f_{q'}(z) - f''(z)| < \min(\frac{1}{2}\varepsilon_{q'}, \frac{1}{2}\delta_0)$.

Let

$$q = \left(h_{q'} \restriction N_\alpha, f'', \frac{1}{2}\varepsilon_{q'}, n_{q'}\right).$$

Claim 3.10. *$q \in P(\alpha)$, q is an extension of q_1 and extends a member of S_R^i.*

Proof. The first statement follows easily from (e). For $q \leqslant q_1$ note that for each $z \in \mathbb{C}^t$ such that $|z| \leqslant n_{q_1} (\leqslant n_{q'})$,

$$\begin{aligned}\left|f_q(z) - f_{q_1}(z)\right| + \varepsilon_q &\leqslant \left|f_q(z) - f_{q'}(z)\right| + \left|f_{q'}(z) - f_{q_1}(z)\right| + \varepsilon_q \\ &= \left|f''(z) - f_{q'}(z)\right| + \left|f_{q'}(z) - f_{q_1}(z)\right| + \frac{1}{2}\varepsilon_{q'} \\ &\leqslant \frac{1}{2}\varepsilon_{q'} + \left|f_{q'}(z) - f_{q_1}(z)\right| + \frac{1}{2}\varepsilon_{q'} \\ &= \left|f_{q'}(z) - f_{q_1}(z)\right| + \varepsilon_{q'} \leqslant \varepsilon_{q_1}.\end{aligned}$$

For the last part of the claim, let $r \in S_R^i$ be such that $q' \leqslant r$. We claim that $q \leqslant r$. Since $h_{q'} \supseteq h_r$ and $h_r \subseteq N_\alpha$, we do have $h_q \supseteq h_r$. Also, for each $z \in \mathbb{C}^t$ such that $|z| \leqslant n_r (\leqslant n_{q'})$, we get $|f_q(z) - f_r(z)| + \varepsilon_q \leqslant \varepsilon_r$ as above, replacing q_1 with r.

This proves Claim 3.10. □

From Claim 3.10 we get

$$q \Vdash_P R \subseteq U_i.$$

For $u \in K'$, choose a rational rectangle $R'_u \subseteq (b_u, e_u)$ so that when $u = d_{x_\alpha}$ we also have $R'_u \subseteq R$.

By 3.9(g) and the fact that $q' \leqslant q_1$, we have for all $z \in \mathbb{C}^t$ such that $|z| \leqslant M (\leqslant n_{q_1} \leqslant n_{q'})$,

$$\left|f''(z) - f_{q_1}(z)\right| \leqslant \left|f''(z) - f_{q'}(z)\right| + \left|f_{q'}(z) - f_{q_1}(z)\right| < \frac{1}{2}\delta_0 + \frac{1}{2}\delta_0 = \delta_0.$$

By the choice of δ_0, it follows that for any rational number μ such that $0 < \mu < \mu'_0$,

$$W\big(f'', \mu, n_1, K, K', \big((b_u, e_u)\colon u \in K'\big)\big) \subseteq B_\delta \subseteq U$$

and hence part 4 of the claim holds. For μ small enough we have that part 5 of the claim holds. (See the proof of part 6 of Claim 3.8.)

This proves Claim 3.9. □

Hence, by Remark 3.7(c), there are such q, μ, R, R'_u for $u \in K'$, for which (modulo identifying $\operatorname{dom}\tilde{h}_1$ with K')

$$\big(\big(x, h_{p_2}(x)\big)\colon x \in \operatorname{dom}\tilde{h}_1\big) \in W\big(f_q, \mu, n_1, K, K', (R'_u\colon u \in K')\big),$$

in particular, $x_\alpha \in R'_{d_{x_\alpha}}$ and the clauses of Claim 3.9 hold. Fix σ_v, for $v \in K'$, such that $|\sigma_v| < \mu$ and $h_{p_2} \subseteq f_{\bar{q}}$, where

$$f_{\bar{q}} = f_q + \sum\big\{\sigma_v Hg\big(n_1, K \setminus \{v\}\big)\colon v \in K'\big\}.$$

Let $\bar{q}$ be the condition

$$\bar{q} = \left(h_q \cup h_{p_2}, f_{\bar{q}}, \frac{1}{2}\varepsilon_q, n_q\right).$$

By part 5 of Claim 3.9, $\bar{q} \leqslant q$. We have by part 2 of Claim 3.9

$$\bar{q} \Vdash_P x_\alpha \in R'_{d_{x_\alpha}} \subseteq R \subseteq \dot{U}_i,$$

which is a contradiction since $\bar{q} \leqslant p_2$ as can be seen by noting that for z such that $|z| \leqslant n_{p_2}$,

$$\begin{aligned}
\left|f_{\bar{q}}(z) - f_{p_2}(z)\right| + \varepsilon_{\bar{q}} &\leqslant \left|f_{\bar{q}}(z) - f_q(z)\right| + \left|f_q(z) - f_{p_2}(z)\right| + \varepsilon_{\bar{q}} \\
&\leqslant \left|f_q(z) - f_{p_2}(z)\right| + \varepsilon_q \\
&\leqslant \left|f_q(z) - f_{q_0}(z)\right| + \left|f_{q_0}(z) - f_{p_2}(z)\right| + \varepsilon_q \\
&\leqslant \left|f_{q_0}(z) - f_{p_2}(z)\right| + \varepsilon_{q_0} \\
&= \left|f'(z) - f_{p_2}(z)\right| + \frac{1}{2}\varepsilon_{p_2} \\
&\leqslant \frac{1}{2}\varepsilon_{p_2} + \frac{1}{2}\varepsilon_{p_2} = \varepsilon_{p_2}. \qquad \square
\end{aligned}$$

4. Proof of Theorem 1.6

Recall the following properties of oracle-cc forcing. See [13, Chapter IV] for more details.

Proposition 4.1. *Assume $\diamondsuit$. Let A be a nonmeager subset of $\mathbb{R}$. Then there is an oracle $\overline{M} = \langle M_\delta\colon \delta < \omega_1\rangle$ such that if P is any partial order satisfying the $\overline{M}$-cc, then $\Vdash_P$ "A is nonmeager".*

Proposition 4.2. *The $\overline{M}$-cc satisfies the following properties.*

(1) *If $\alpha < \omega_2$ is a limit ordinal, $\langle\langle P_\beta\rangle_{\beta\leqslant\alpha}, \langle\dot{Q}_\beta\rangle_{\beta<\alpha}\rangle$ is a finite-support α-stage iteration of partial orders, and for each $\beta < \alpha$, P_β is $\overline{M}$-cc, then P_α is $\overline{M}$-cc.*
(2) *If P is $\overline{M}$-cc, then there is a P-name $\overline{M}^*$ for an oracle such that for each P-name $\dot{Q}$ for a partial order, if $\Vdash_P$ "$\dot{Q}$ is $\overline{M}^*$-cc" then $P * \dot{Q}$ is $\overline{M}$-cc.*
(3) *If $\overline{M}_\alpha$, $\alpha < \omega_1$, are oracles, then there is an oracle $\overline{M}$ such that for any partial order P, if P is $\overline{M}$-cc, then P is $\overline{M}_\alpha$-cc for all $\alpha < \omega_1$.*

Proof of Theorem 1.6. . First note that by approximating g using Proposition 1.8, we may assume that g is entire (more precisely, the restriction to $\mathbb{R}^n$ of a entire function taking real values on $\mathbb{R}^n$). Second, if g is entire, then it is C^∞, so (c)(i) need not be considered as it is a special case of (c)(ii). Third, we may assume that $g \equiv 0$. To see this, let $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the homeomorphism defined by

$$T(x, y) = \bigl(x, y + g(x)\bigr), \quad x \in \mathbb{R}^n,\ y \in \mathbb{R}.$$

Replace $\mathcal{C}$ by $\{T^{-1}[C]\colon C \in \mathcal{C}\}$ and apply the case of Theorem 1.6 where $g \equiv 0$. This produces an entire function f satisfying $|D^\alpha f(x)| < \varepsilon(x)$ for the appropriate values of α and x and such that $T^{-1}[C] \cap f$ is everywhere second category in f for each $C \in \mathcal{C}$. Then $C \cap (f + g) = T[T^{-1}[C] \cap f]$ is everywhere second category in $f + g$, $f + g$ is (the restriction to $\mathbb{R}^n$ of) an entire function and $|D^\alpha(f + g)(x) - D^\alpha g(x)| = |D^\alpha f(x)| < \varepsilon(x)$ for the appropriate values of α and x. For (c)(iii), leave A as is and replace B_x by $B_x - g(x)\{y - g(x)\colon y \in B_x\}$ for each $x \in A$. Note that if $f(x) \in B_x - g(x)$ then $(f + g)(x) \in B_x$.

The rest of the proof is standard oracle-cc technique. We sketch the argument. The argument is similar to the one in [2, Section 5]. Start with a ground model satisfying $V = L$. Fix a diamond sequence

$$\bigl\langle(x_\alpha, t_\alpha, a_\alpha, b_\alpha, d_\alpha, c_\alpha, e_\alpha)\colon \alpha < \omega_2,\ \operatorname{cof}(\alpha) = \omega_1\bigr\rangle$$

for trapping septuples (x, t, a, b, d, c, e) consisting of:

(1) A function $x : \omega_2 \to ([\omega_2]^{\leqslant\omega})^\omega$. The idea of x is that, with ω_2 identified with the ccc partial order we are about to build, $[\omega_2]^{\leqslant\omega}$ contains the antichains. Thus, $([\omega_2]^{\leqslant\omega})^\omega$ contains a name for each real number (construed as

a subset of ω). Then for any second category set $X \subseteq \mathbb{R}$ in the extension, we can find a ground model function $x : \omega_2 \to ([\omega_2]^{\leqslant\omega})^\omega$ enumerating the names of the elements of X.

(2) $t \in ([\omega_2]^{\leqslant\omega})^\omega$ intended to represent a name for a natural number (the number of variables).

(3) Functions $a : \omega \to ([\omega_2]^{\leqslant\omega})^\omega$, $b : \omega \times \omega \to ([\omega_2]^{\leqslant\omega})^\omega$ and $d : \omega_1 \times \omega_1 \to ([\omega_2]^{\leqslant\omega})^\omega$ representing respectively (enumerations of the names for the elements of) a dense subset $\{a(n)\colon n < \omega\}$ of $\mathbb{R}^t$, dense subsets $\{b(i,n)\colon n < \omega\}$, $i < \omega$, of $\mathbb{R}$, and everywhere second category subsets of $\mathbb{R}^{t+1}$ of cardinality $\aleph_1$, $\{d(\alpha,\xi)\colon \xi < \omega_1\}$, $\alpha < \omega_1$.

(4) A function $c : \omega \to ([\omega_2]^{\leqslant\omega})^\omega$ so that $\langle c(i)\colon i < \omega\rangle$ represents a sequence of names for the terms of a nondecreasing sequence of nonnegative real numbers converging to ∞.

(5) A function $e \in ([\omega_2]^{\leqslant\omega})^\omega$ intended to represent a name for the Borel code of a positive continuous function.

So for each $\alpha < \omega_2$ of cofinality ω_1, we have $x_\alpha : \alpha \to ([\alpha]^{\leqslant\omega})^\omega$, $b_\alpha : \omega \times \omega \to ([\alpha]^{\leqslant\omega})^\omega$, $d_\alpha : \omega_1 \times \omega_1 \to ([\alpha]^{\leqslant\omega})^\omega$, $t_\alpha, e_\alpha \in ([\alpha]^{\leqslant\omega})^\omega$ and $a_\alpha, c_\alpha : \omega \to ([\alpha]^{\leqslant\omega})^\omega$. Also, for each (x,t,a,b,d,c,e) as in (1)–(5), $\{\alpha < \omega_2\colon \operatorname{cof}(\alpha) = \omega_1,\ x \restriction \alpha = x_\alpha,\ t = t_\alpha,\ a = a_\alpha,\ b = b_\alpha,\ d = d_\alpha,\ c = c_\alpha$ and $e = e_\alpha\}$ is stationary in ω_2.

We will inductively define an ω_2-stage finite support iteration

$$\big\langle \langle P_\alpha\rangle_{\alpha\leqslant\omega_2}, \langle \dot{Q}_\alpha\rangle_{\alpha<\omega_1}\big\rangle$$

as well as a P_α-names $\overline{M}_\alpha$ for oracles and one-to-one functions $F_\alpha : P_\alpha \to \omega_2$ for $\alpha < \omega_2$ such that the range of each F_α is an initial segment of ω_2 which includes α and for $\beta < \alpha < \omega_2$, we have $F_\beta \subseteq F_\alpha$. (At each stage, F_α is any function satisfying these conditions.)

For $\alpha < \omega_2$, we make the following definitions after P_α and F_α are defined.

(6) $\dot{X}_\alpha$ denotes the P_α-name for the set of real numbers whose elements have the names

$$\bigcup_{n<\omega} \{n\} \times F_\alpha^{-1}\big[x_\alpha(\xi)(n)\big], \quad \xi < \alpha.$$

(7) For each $i < \omega$, $\dot{B}_{\alpha i}$ denotes the ω-sequences of P_α-names for real numbers

$$\Big\langle \bigcup_{n<\omega} \{n\} \times F_\alpha^{-1}\big[b_\alpha(i,j)(n)\big]\colon j < \omega \Big\rangle$$

and for each $\eta < \omega_1$, $\dot{C}_{\alpha\eta}$ denotes the ω_1-sequences of P_α-names for real numbers

$$\Big\langle \bigcup_{n<\omega} \{n\} \times F_\alpha^{-1}\big[d_\alpha(\eta,\xi)(n)\big]\colon \xi < \omega_1 \Big\rangle.$$

(8) $\dot{A}_\alpha$ and $\dot{c}_\alpha$ denote the ω-sequences of P_α-names for elements of $\mathbb{R}^t$ and $\mathbb{R}$, respectively, given, respectively, by

$$\Big\langle \bigcup_{n<\omega} \{n\} \times F_\alpha^{-1}\big[a_\alpha(i)(n)\big]\colon i < \omega \Big\rangle \quad \text{and} \quad \Big\langle \bigcup_{n<\omega} \{n\} \times F_\alpha^{-1}\big[c_\alpha(i)(n)\big]\colon i < \omega \Big\rangle.$$

(9) $\dot{t}_\alpha$ and $\dot{e}_\alpha$ denote respectively the P_α-names for real numbers given by

$$\bigcup_{n<\omega} \{n\} \times F_\alpha^{-1}\big[t_\alpha(n)\big] \quad \text{and} \quad \bigcup_{n<\omega} \{n\} \times F_\alpha^{-1}\big[e_\alpha(n)\big].$$

At stage $\alpha < \omega_2$ of the construction, if $\operatorname{cof}(\alpha) = \omega_1$ and if

$$\Vdash_{P_\alpha} \dot{X}_\alpha \quad \text{is second category,}$$

then we use Lemma 4.1 to get a P_α-name $\overline{M}'_\alpha$ for an oracle so that if P is any forcing notion which satisfies the $\overline{M}'_\alpha$-cc, then X_α remains second category after forcing with P. Otherwise, in particular if $\operatorname{cof}(\alpha) \neq \omega_1$, we let $\overline{M}'_\alpha$ be any P_α-name for an oracle.

For $\beta < \alpha$, let $P_{\beta\alpha}$ be the usual P_β-name for a partial order such that P_α is isomorphic to a dense subset of $P_\beta * P_{\beta\alpha}$. Let $\overline{M}_{\beta\alpha}$ be a P_α-name for an oracle such that

(10) If $\Vdash_{P_\beta}$ "$P_{\beta,\alpha}$ is $\overline{M}_\beta$-cc and $\Vdash_{P_{\beta,\alpha}} \dot{Q}_\alpha$ is $\overline{M}_{\beta\alpha}$-cc", then $\Vdash_{P_\beta}$ "$P_{\beta,\alpha+1} = P_{\beta,\alpha} * \dot{Q}_\alpha$ is $\overline{M}_\beta$-cc".

Now, if $\mathrm{cof}(\alpha) = \omega_1$ and if

(11) $\Vdash_{P_\alpha} \dot{t}_\alpha$ is a positive integer,
(12) $\Vdash_{P_\alpha}$ The range of $\dot{A}_\alpha$ is dense in $\mathbb{R}^{\dot{t}_\alpha}$ and for all $i < \omega$, $\dot{B}_{\alpha i}$ is dense in $\mathbb{R}$,
(13) $\Vdash_{P_\alpha}$ For all $\eta < \omega_1$, the range of $\dot{C}_{\alpha\eta}$ is everywhere second category in $\mathbb{R}^{\dot{t}_\alpha+1}$,
(14) $\Vdash_{P_\alpha} \dot{c}_\alpha(0) \leqslant \dot{c}_\alpha(1) \leqslant \cdots$ and $\lim_{i\to\infty} \dot{c}_\alpha(i) = \infty$, and
(15) $\Vdash_{P_\alpha} \dot{e}_\alpha$ is a positive continuous function,

then use Lemma 3.5 to get a P_α-name $\dot{Q}_\alpha$ for a partial order satisfying the $\overline{M}_\alpha$-cc (see (17) below for the definition) and forcing an entire function defining an everywhere second category section through each $\dot{C}_{\alpha\eta}$, as described in the statement of the lemma. (For $i < \omega$, corresponding to $x = \dot{A}(i)$, take $B_x = \dot{B}_{\alpha i}$.) Then we let $\overline{M}^*_{\alpha+1}$ be a $P_{\alpha+1}$-name for an oracle such that $\overline{M}^*_{\alpha+1}$-cc forcing preserves the everywhere second category nature of the sections. In all other cases, take $\dot{Q}_\alpha$ to name the partial order Q for adding one Cohen real. We have thus

(16) $\Vdash_{P_\alpha}$ "$\dot{Q}_\alpha$ satisfies the $\overline{M}_\alpha$-cc".

Let $\overline{M}_\alpha$ be a P_α-name for an oracle such that

(17) $\Vdash_{P_\alpha}$ "If $\dot{Q}_\alpha$ is $\overline{M}_\alpha$-cc, then $\dot{Q}_\alpha$ is $\overline{M}'_\alpha$-cc and (if $\overline{M}^*_\alpha$ is defined) $\overline{M}^*_\alpha$-cc and $\overline{M}_{\beta\alpha}$-cc for all $\beta < \alpha$".

Now suppose that for some P_{ω_2}-name $\dot{X}$ we have $\Vdash_{P_{\omega_2}} \dot{X}$ is second category. Fix a name $\dot{x}$ such that $\Vdash_{P_{\omega_2}} \dot{x} : \omega_2 \to \dot{X}$ is onto. Then define $x : \omega_2 \to ([\omega_2]^{\leqslant\omega})^\omega$ so that if

$$\tau_\xi = \bigcup_{n<\omega} \{n\} \times F^{-1}\big[x(\xi)(n)\big], \quad \xi < \omega_2,$$

then for each $\xi < \omega_2$, $\Vdash_{P_{\omega_2}} \dot{x}(\xi) = \tau_\xi$. There is a closed unbounded set $C \subseteq \omega_2$ such that for each $\alpha \in C$ of cofinality ω_1 we have:

(18) $x \restriction \alpha : \alpha \to ([\alpha]^{\leqslant\omega})^\omega$,
(19) $\forall \xi < \alpha$, τ_ξ is a P_α-name,
(20) $\Vdash_{P_\alpha} \{\tau_\xi \colon \xi < \alpha\}$ is second category.

Choose such an α of cofinality ω_1 for which $x \restriction \alpha = x_\alpha$. By (18) and (19), the definition of τ_ξ for $\xi < \alpha$ would not change if we used x_α instead of x and F_α instead of F. Then from the definition of $\dot{X}_\alpha$ we get

$$\Vdash_{P_\alpha} \dot{X}_\alpha = \{\tau_\xi \colon \xi < \alpha\}.$$

So at stage α we chose a P_α-name $\overline{M}_\alpha$ and we arranged that $\Vdash_{P_\alpha}$ "$P_{\alpha,\gamma}$ is $\overline{M}_\alpha$-cc. (This follow by induction on $\gamma \geqslant \alpha$ using Proposition 4.2(1) at limits, and using (16), (17) and (10) above at successors.) Hence, by the choice of $\overline{M}_\alpha$, $\Vdash_{P_\alpha}\Vdash_{P_{\alpha,\gamma}}$ "$\dot{X}_\alpha$ is second category" from which it follows that $\Vdash_{P_\alpha}\Vdash_{P_{\alpha,\omega_2}}$ "$\dot{X}_\alpha$ is second category".

By what we have established, there are guaranteed to be sets of cardinality ω_1 which are second category in any extension by P_{ω_2}. Hence there are guaranteed to be everywhere second category sets of cardinality ω_1. Suppose that for some P_{ω_2}-names $\dot{t}$, $\dot{A}$, $\dot{B}_i$ for $i < \omega$, $\dot{c}_i$ for $i < \omega$, and $\dot{f}$, $\dot{e}$ we have

(21) $\Vdash_{P_{\omega_2}} \dot{t}$ is a positive integer,
(22) $\Vdash_{P_{\omega_2}} \dot{A} : \omega \to \mathbb{R}^{\dot{t}}$ has dense range and for all $i < \omega$, $\dot{B}_i$ is dense in $\mathbb{R}$,
(23) $\Vdash_{P_{\omega_2}}$ for all $\eta < \omega_1$, $\dot{C}_\eta$ is an everywhere second category subset of $\mathbb{R}^{\dot{t}+1}$,
(24) $\Vdash_{P_{\omega_2}} \dot{c}_0 \leqslant \dot{c}_1 \leqslant \cdots$ and $\lim_{i\to\infty} \dot{c}_i = \infty$, and
(25) $\Vdash_{P_{\omega_2}} \dot{e}$ is a positive continuous function.

Define t, a, b, d, c, e to be functions as in (2)–(5) above so that letting

(26) $\sigma_i^0 = \bigcup_{n<\omega}\{n\} \times F^{-1}[a(i)(n)]$ for $i < \omega$, $\sigma_{ij}^1 = \bigcup_{n<\omega}\{n\} \times F^{-1}[b(i,j)(n)]$ for $i, j < \omega$ and $\tau_{\eta\xi} = \bigcup_{n<\omega}\{n\} \times F^{-1}[d(\eta,\xi)(n)]$ for $\eta, \xi < \omega_1$, we have for all $i, j < \omega$ and $\eta, \xi < \omega_1$, $\Vdash_{P_{\omega_2}} \dot{A}(i) = \sigma_i^0$, $\Vdash_{P_{\omega_2}} \dot{B}_i(j) = \sigma_{ij}^1$ and $\Vdash_{P_{\omega_2}} \dot{C}_\eta(\xi) = \tau_{\eta\xi}$,

(27) $\gamma_i = \bigcup_{n<\omega}\{n\} \times F^{-1}[c(i)(n)]$, $i < \omega$, we have for each $i < \omega$, $\Vdash_{P_{\omega_2}} \dot{c}(i) = \gamma_i$,

(28) $\varphi = \bigcup_{n<\omega}\{n\} \times F^{-1}[t(n)]$ and $\varepsilon = \bigcup_{n<\omega}\{n\} \times F^{-1}[e(n)]$, we have $\Vdash_{P_{\omega_2}} \dot{t} = \varphi$ and $\Vdash_{P_{\omega_2}} \dot{e} = \varepsilon$.

For all large enough $\alpha < \omega_2$, we have:

(29) $b : \omega \times \omega \to ([\alpha]^{\leqslant\omega})^\omega$, $d : \omega_1 \times \omega_1 \to ([\alpha]^{\leqslant\omega})^\omega$, $a, c : \omega \to ([\alpha]^{\leqslant\omega})^\omega$, and $t, e : \omega \to [\alpha]^{\leqslant\omega}$,

(30) for all values of the indices for which they are defined, σ_i^0, σ_{ij}^1, $\tau_{\eta\xi}$, γ_i, φ and ε are P_α-names.

Choose any such α of cofinality ω_1 for which $(t, a, b, d, c, e) = (t_\alpha, a_\alpha, b_\alpha, d_\alpha, c_\alpha, e_\alpha)$. By (29) and (30), the definitions of σ_i^0, σ_{ij}^1, $\tau_{\eta\xi}$, γ_i, φ and ε would not change if we used $t_\alpha, a_\alpha, b_\alpha, d_\alpha, c_\alpha, e_\alpha$ instead of t, a, b, d, c, e, respectively, and F_α instead of F. Then from the definitions of $\dot{t}_\alpha$, $\dot{A}_\alpha$, $\dot{B}_{\alpha i}$ for $i < \omega$, $\dot{C}_{\alpha\eta}$ for $\eta < \omega_1$, $\dot{c}_\alpha$, and $\dot{e}_\alpha$, we get that (11)–(15) hold. (Being everywhere second category is trivially downward absolute.) Then $\dot{Q}_\alpha$ was chosen to add an entire function f of the desired type. Most of its properties are upward absolute. The property that the sections determined by f are everywhere second category in f is preserved because $P_{\alpha+1,\gamma}$ is $\overline{M}^*_{\alpha+1}$-cc for every $\gamma < \omega_1$.

This completes the proof of the theorem. □

Acknowledgements

We thank Ed Bierstone, Ilijas Farah, Paul Gauthier, Ian Graham, Juris Steprans, Stevo Todorcevic and Bill Weiss for helpful discussions and Krzysztof Ciesielski for correspondence concerning [4]. This paper was motivated by the work on [3].

References

[1] T. Bartoszynski, L. Halbeisen, There are big symmetric planar sets meeting polynomials at just finitely many points, Preprint.
[2] M.R. Burke, Entire functions mapping uncountable dense sets of reals onto each other monotonically, submitted for publication.
[3] M.R. Burke, A.W. Miller, Models in which every nonmeager set is nonmeager in a nowhere dense Cantor set, Canad. J. Math. 57 (2005) 1139–1154.
[4] K. Ciesielski, T. Natkaniec, A big symmetric planar set with small category projections, Fund. Math. 178 (2003) 237–253.
[5] K. Ciesielski, S. Shelah, Category analogue of sup-measurability problem, J. Appl. Anal. 6 (2000) 159–172.
[6] E.M. Frih, P. Gauthier, Approximation of a function and its derivatives by entire functions of several variables, Canad. Math. Bull. 31 (1988) 495–499.
[7] M. Hervé, Several Complex Variables; Local Theory, Oxford University Press, Oxford, 1963.
[8] L. Hoischen, Eine Verschärfung eines approximationssatzes von Carleman, J. Approximation Theory 9 (1973) 272–277.
[9] K. Kunen, Set Theory, North-Holland, Amsterdam, 1983.
[10] C. Kuratowski, S. Ulam, Quelques propriétés topologiques du produit combinatoire, Fund. Math. 19 (1932) 247–251.
[11] J.C. Oxtoby, Measure and Category, second ed., Springer, Berlin, 1980.
[12] S. Shelah, Independence results, J. Symbolic Logic 45 (1980) 563–573.
[13] S. Shelah, Proper and Improper Forcing, second ed., Springer, Berlin, 1998.